THE

———

XVIII. *Temperature-Radiation and the Partition of Energy in Continuous Media. By* J. H. JEANS, *M.A., F.R.S., Professor of Applied Mathematics in Princeton University*.*

CONTENTS.

INTRODUCTION.

1. THE present paper attempts a critical examination of some of the difficulties which surround the application of the law of equipartition of energy to the energy of wave motion in the æther or other continuous media. These difficulties manifest themselves most clearly in connexion with the problem of determining the law of partition of radiant energy inside a uniform-temperature enclosure. Accordingly it is with this particular problem that we shall be mainly concerned.

2. Let two or more bodies, originally at different temperatures, be supposed placed inside a closed chamber whose

walls reflect energy perfectly. These conditions cannot be realized experimentally; so that it is not known what would happen. But it is commonly supposed that the temperatures of the bodies would equalize * by radiation, so that after an infinite time the bodies would all be at the same temperature T; also the æther, the vehicle of energy from one body to another, would be possessed of a certain amount of energy, and in spite of the continual transference and retransference of energy between matter and æther, the amount of energy in the æther, and the law of its distribution between different wave-lengths, would remain constant. In this supposed state, let the energy per unit volume of radiation of wave-lengths intermediate between λ and $\lambda + d\lambda$ be assumed to be

$$F(\lambda, T)d\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

3. The position of the masses of matter inside the enclosure has been immaterial. Let them now be supposed spread over the walls, so that every part of the original perfectly-reflecting walls is covered. So far the whole system inside the reflecting walls has remained impervious to energy. Let a minute hole now be made at any point in one of the walls, and let the radiant energy stream through this hole into external space (which may, for simplicity, be thought of as devoid of radiant energy). At first the issuing stream of radiant energy will be of constitution given by formula (1); but in time, as the total amount of energy inside the enclosure diminishes, the constitution of the escaping energy will change. The flow of energy could be kept constant, provided energy of the appropriate amount *and constitution* could be supplied to the inside of the enclosure. There would then be a steady flow of energy through the aperture, of constitution given by formula (1).

4. Suppose that the walls are no longer perfectly-reflecting, but are provided with a mechanism which keeps them at a uniform temperature T. The flow of energy through the small aperture must now be " steady ": experiment shows it to be independent of the nature and reflecting power of the

---

* This is the orthodox view. Equalization is commonly supposed to be a necessity, from the second law of thermodynamics. My own view is that equalization *would take place*, but only because the charges on the electric sources of radiation would be identical in the different masses of matter. If the electrons in one mass were more heavily charged than those in the other masses, I believe the temperatures would tend to become different, in opposition to the second law. (*Cf.* Phil. Mag. [6] xii. p. 57.)

walls. Let the constitution of this stream of energy be supposed given by the formula

$$f(\lambda, \text{ T})d\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It is commonly argued or assumed * that the constitution of the energy in this experiment must be the same as in the preceding imaginary experiment. If so, the functions F and f must be identical.

5. Assuming the legitimacy of using the theorem of Equipartition of Energy, the function F can be calculated at once from this theorem. It is found † that

$$\text{F}(\lambda, \text{ T}) = 8\pi \text{RT}\lambda^{-4}, \quad . \quad . \quad . \quad . \quad (3)$$

where R is the universal gas-constant.

The function f can be determined experimentally. Planck ‡, with the help of a mathematical argument, with the details of which we are not here concerned, arrives at the formula

$$f(\lambda, \text{ T}) = \frac{8\pi c}{\lambda^5} \frac{1}{e^{\frac{c}{k\text{T}\lambda}} - 1}, \quad . \quad . \quad . \quad (4)$$

a form which agrees well § with experimental readings, provided c and k may be treated as adjustable constants.

The values of c and k which Planck arrives at by comparing formula (4) with experiment are

$$c = 1\cdot965 \times 10^{-16}, \qquad k = 1\cdot346 \times 10^{-16}.$$

On the other hand, Lorentz ‖, using the form $8\pi k\lambda^{-4}\text{T}$, to which formula (4) reduces for long wave-lengths, obtains

$$k = 1\cdot07 \times 10^{-16}$$

as the value of k given by experiments on light of great wave-length. According to Planck's analysis, the constant k ought to be identical with the gas constant R, of which the value is known to be $\text{R} = 10^{-16}$ to within a few per cent.

---

* The assumption is tacitly involved in the common employment of the expression "radiation appropriate to a given temperature." (*Cf.* Proc. Roy. Soc. A. lxxvi. p. 306, 1905.)

† Rayleigh, Phil. Mag. [5] xlix. p. 539 (1900), and Nature, lxxii. pp. 54, 243 (1905). Also J. H. Jeans, Phil. Mag. [6] x. p. 91 (1905).

‡ *Vorlesungen über Wärmestrahlungen* (1906), p. 157, and in earlier papers.

§ Planck, *Vorlesungen über Wärmestrahlungen*, p. 158, and Paschen, *Annalen d. Physik*, iv. p. 277.

‖ *Konink. Akad. van Wetenschappen* (Amsterdam), April 24, 1903, p. 678.

Let us, for the present, continue to denote R and k by separate symbols, leaving open the question of whether or not the quantities they represent are identical. If these quantities are not identical, then the formulæ (3) and (4) are entirely different.   But if they are found to be identical, then the formulæ will be seen to coincide for light of great wave-length, but to diverge widely for light of visible or very short wave-length.

6. The divergence between the two formulæ, whether complete or partial, raises various questions, to which the analysis of the present paper attempts to provide answers. It is natural to inquire—

(1) Is the use of the Theorem of Equipartition, and consequent derivation of formula (3), legitimate?

(2) If so, what is the essential difference between the physical conditions which lead to formula (3) and those which lead to formula (4)?

(3) If equation (1) is answered in the affirmative, do formulæ (3) and (4) become identical for long wave-lengths; and if so, why?

It may simplify what follows to state briefly in advance the conclusions arrived at.

7. It is found that question (1) can be definitely answered in the affirmative, but that the theorem of equipartition represents merely the tendency for energy to become degraded into irregular disturbances of the medium, and the utility of the theorem of equipartition (although not, of course, its truth) is limited by the circumstance that it represents a state attained only after enormous, or infinite, time.

The answer obtained to question (2) can be best explained by making use of an acoustical analogy.   Let the æther be replaced by air : let waves of light in the æther be represented by waves of sound in air.   Matter may be represented by a series of musical or noise-producing instruments : these will of course be capable of absorbing as well as emitting sound.

To represent the state of things considered in § 4, we have to imagine the walls of a room to be covered continuously with sound-instruments (to make the picture clearer, let us say telephone diaphragms), and we suppose that these are kept in vibration by agencies acting from outside the room. A person listening at an aperture in the wall of the room will hear the sound of the telephones (modified by the reflexion and absorption of the other diaphragms, perhaps): the

energy of these sounds corresponds to that given in formula (4).

To represent the state of things imagined in § 3, suppose that the walls behind the diaphragms are impervious to sound and to all kinds of energy, and that the aperture in the wall is stopped up with matter also impervious to energy. The diaphragms must be supposed to be initially set into vibration, and then left to themselves. The energy of the sounds they emit will be reflected, absorbed, re-emitted, and so on. Finally all sound will have become dissipated into heat. The energy of the original sounds—the total store of energy in the room—will figure as the heat of the air and of the diaphragms. A person outside the room who uncovers the aperture and listens will hear nothing at all, unless his ears are sufficiently acute to hear the waves of air originating in the random heat-motions of the molecules. For, as we shall see, this random motion of the molecules can be resolved, by Fourier's theorem, into the motion of trains of waves, and, in perfectly irregular heat motion, there is equipartition of energy between the different trains of waves, so that the law of partition according to wave-lengths is given by the formula $4\pi RT\lambda^{-4}d\lambda$. This agrees with formula (3), except for a numerical factor 2 ˙which finds its origin in the different energy-capacities of transverse and longitudinal vibrations.

It is found that the third question must be answered in the affirmative. The reason for this answer will be found in the concluding sections (§§ 22–30), and the reader who is not interested in the abstract argument and analysis which follow is advised to pass at once to these sections.

The "Normal State" and Equipartition of Energy.

8. We begin the mathematical discussion by proving the law of equipartition in the form appropriate to the vibrational energy of continuous media. It is important to exhibit the proof in such a form as to make it clear that *the law rests on no assumptions of any kind, except the assumption that the motion of the medium obeys the laws of a conservative dynamical system.*

Let L be a function of variables $\theta_1, \theta_2, \ldots \theta_n, \dot{\theta}_1, \dot{\theta}_2, \ldots \dot{\theta}_n$, and let these change in value so that $\dfrac{d\theta_1}{dt} = \dot{\theta}_1$, &c., while $\int L dt$ is stationary in value. Let L consist solely of terms of degrees two and zero in $\dot{\theta}_1, \dot{\theta}_2, \ldots \dot{\theta}_n$.

New functions $u_1, u_2, \ldots u_n$ and L' of these variables are introduced, defined by

$$u_1 = \frac{\partial L}{\partial \dot{\theta}_1}, \quad \&c.$$

$$L' = u_1 \dot{\theta}_1 + u_2 \dot{\theta}_2 + \ldots + u_n \dot{\theta}_n - L;$$

and it is then proved in the usual way, by purely algebraic transformation *, that

$$\frac{du_1}{dt} = -\frac{\partial L'}{\partial \theta_1}; \quad \frac{d\theta_1}{dt} = \frac{\partial L'}{\partial u_1}; \quad \cdots \quad (5)$$

whence it follows that

$$\frac{\partial}{\partial u_1}\left(\frac{du_1}{dt}\right) + \frac{\partial}{\partial \theta_1}\left(\frac{d\theta_1}{dt}\right) = 0. \quad \cdots \quad (6)$$

The condition that $\int L \, dt$ is to be stationary determines uniquely the changes in $\theta_1, \theta_2, \ldots \theta_n, \dot{\theta}_1, \dot{\theta}_2, \ldots \dot{\theta}_n$, starting from given initial values. Now let a generalized space be constructed, having $\theta_1, \theta_2, \ldots \theta_n, u_1, u_2 \ldots u_n$, as its coordinates. Let this space be filled with " representative points " each of which is to move as directed by the condition that $\int L \, dt$ is to be stationary. If ρ denote the density of these " representative points " at any point in the generalized space, and if $\frac{D\rho}{Dt}$ denotes the rate of increase of ρ as we follow the " representative points " in their motion, we have †, as a matter of algebraic calculation,

$$\frac{D\rho}{Dt} = -\rho \sum_1^n \left\{ \frac{\partial}{\partial u_1}\left(\frac{du_1}{dt}\right) + \frac{\partial}{\partial \theta_1}\left(\frac{d\theta_1}{dt}\right) \right\} = 0, \quad \cdot \quad (7)$$

by equation (6). If the " representative points " tended, in their motion, to concentrate onto any special points or regions in the generalized space, then $\frac{D\rho}{Dt}$ would be positive for those points or regions : similarly, if the " representative points " tended to scatter, $\frac{D\rho}{Dt}$ would be negative. The result obtained in equation (7), that $\frac{D\rho}{Dt}$ vanishes everywhere, shows that there is no tendency for the " representative points " to

* Routh, ' Elem. Rigid Dynamics,' chap. viii., or Jeans, ' Theoretical Mechanics,' chap. xii.
† Jeans, ' Dynamical Theory of Gases,' p. 63.

concentrate about any special points or regions in the generalized space, or the reverse.

We have so far merely studied the changes in the values of a system of algebraic variables, when they change as directed by a certain algebraic system of conditions (namely, $\delta \int L dt = 0$). The motion of points in the generalized space has merely provided a graphical representation of these changes.

9. Now let these variables $\theta_1, \theta_2, \ldots \theta_n, \dot{\theta}_1, \dot{\theta}_2, \ldots \dot{\theta}_n$ be the coordinates and velocities of a dynamical system, and let L be its Lagrangian function. Then the motion of a " representative point " in the generalized space will represent the changes in the coordinates and velocities as these change in accordance with the principle of Least Action—*i. e.*, as the system moves in accordance with the laws of nature. The proved fact that in this motion there is no tendency for the " representative points " to concentrate about any special points or regions of the generalized space leads at once to the following :—

THEOREM. *All properties (if any) which are such as to be finally acquired by the dynamical system, independently of the special state from which the system started, must be properties common to the whole of the generalized space.*

For, if the system must inevitably possess some property, this can only be either because its representative point tends inevitably to pass into the regions of the generalized space in which this property holds, or else because the property holds in all regions. The former alternative is disproved by equation (7) : the latter alternative must accordingly be the true one.

10. It is found that, in general, there are no properties common to all regions in the generalized space (or rather, no properties of any importance for the present purpose). But when the system possesses an infinite (or very great) number of similar coordinates, there are certain statistical properties found to be common to the whole of the space except for infinitesimal regions of it. A system which possesses these statistical properties is said to be in the " Normal State."

A representative point may of course have its whole path in regions in which the " Normal State " does not obtain, or it may pass through these regions for periods, large or small, on its path. But we have the quite general theorem:—

THEOREM. *If a system tends to acquire definite properties, independently of its initial configuration, or if it tends to acquire*

these properties when it starts from any configurations except an infinitesimal few, then these properties must be properties of the "Normal State."

For, if not. suppose that the system tends to assume some property P which is not common to the whole of the generalized space, or even to the whole except for infinitesimal regions, but is confined to some region S of the generalized space. The "representative points" which at the beginning of the motion in the generalized space occupied the whole of this space (or the whole of it except for infinitesimal regions), must, by the end of the motion, all lie within the region S— a result which would be in opposition to equation (7),

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0.$$

11. Of the properties of the "normal state," that one which is of primary importance for the present investigation is the Equipartition of Energy *.

Suppose that the energy E of the system can be expressed as a function of the Lagrangian coordinates and velocities in the form

$$2\mathrm{E} = \alpha_1 p_1^2 + \alpha_2 p_2^2 + \ldots + \alpha_n p_n^2 + f(\theta_1, \theta_2, \ldots \theta_n),$$

where $p_1, p_2, \ldots p_n$ are any quantities (coordinates, velocities, or momenta) and $\theta_1, \theta_2, \ldots \theta_n$ are other quantities which may or may not enter into $\alpha_1, \alpha_2, \ldots \alpha_n$. The law of equipartition states that if n is very great, the energy represented by any very great number s of the n first terms is, in the normal state, proportional to s. We assume it to be $\frac{1}{2}s\mathrm{RT}$. Then, if part of the dynamical system consists of matter of any kind, T is the temperature of this matter †.

12. If the system is supposed, for the moment, to consist solely of a non-dissipative vibrating medium, free from disturbance by external agencies, the whole of the energy can be expressed in the form

$$2\mathrm{E} = \alpha_1 p_1^2 + \alpha_2 p_2^2 + \ldots + \alpha_n p_n^2,$$

in which $p_1, p_2, \ldots p_n$ represent the normal coordinates and their rates of change. Thus each separate free vibration contributes two terms to the energy. In the normal state, the energy of any great number s of free vibrations must be $s\mathrm{RT}$.

13. From a consideration of physical dimensions, it is clear that in any medium whatever, the number of free vibrations

---

* The proof that this is a property of the Normal State is purely algebraic in its nature: see 'The Dynamical Theory of Gases,' p. 67.

† Jeans, 'Dynamical Theory of Gases,' §§ 77, 124.

of wave-lengths intermediate between λ and $\lambda + d\lambda$ (where λ is large compared with the scale of structure of the medium, if the medium is coarse-grained) must be

$$C\lambda^{-4}d\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

per unit volume, where **C** is a numerical quantity which depends on the structure of the medium. It follows that when the medium is in equilibrium with matter at temperature T, its vibrational energy of wave-lengths intermediate between λ and $\lambda + d\lambda$ must be

$$CRT\lambda^{-4}d\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

per unit volume.

If the medium is æther, it is easily found (§ 5) that the value of **C** must be 8π, and formula (3) follows at once. If the medium is gaseous (so that all vibrations are longitudinal) the value of **C** is 4π, while for an elastic solid medium $C = 12\pi$.

14. If the medium is structureless, then formula (9) holds down to the very shortest wave-lengths. The energy corresponding to any finite value of T is infinite. Whatever the value of T, the whole energy (except for an infinitesimal fraction) is confined to vibrations of infinitesimal wave-length. In this case the value of the Law of Equipartition is not so much that it gives the final state of the medium (a state reached only after infinite time (*cf.* § 24, below)), as that it shows the tendency for the energy to run into vibrations of infinitesimal wave-length. Or, what is the same thing, it shows the tendency for regular trains of waves to become dissipated into irregular disturbances (subject to a certain limitation, *cf.* below).

15. If the medium is coarse-grained, then formula (9) is not applicable, when dealing with waves of length comparable with the scale of structure of the medium. Let λ_0 denote (loosely speaking) the smallest wave-length possible, so that λ_0 is a length comparable with the scale of coarse-grainedness of the medium. From formula (9), the total energy per unit volume of the medium is

$$\int_{\lambda = \lambda_0}^{\lambda = \infty} CRT\lambda^{-4}d\lambda = \frac{CRT}{3\lambda_0^3}.$$

If J is the mechanical equivalent of heat, the "specific heat" of the medium must be

$$\frac{CR}{3J\lambda_0^3}$$

per unit volume. Denoting this specific heat by σ, and

replacing R/J by its known value 2×10^{-24}, we find that λ_0 must be of the order of $1 \cdot 3 \times 10^{-8} \times \sigma^{-\frac{1}{3}}$. For instance, for water $\sigma = 1$: it follows that the mean distance apart of the molecules of water must be comparable with $1 \cdot 3 \times 10^{-8}$ cm. For air at atmospheric pressure, $\sigma = \cdot 0002$: it follows that the mean distance apart of the molecules of the atmosphere must be comparable with 2×10^{-7} cm. If it were possible to measure the energy of æther in temperature-equilibrium with matter, we could determine the specific heat σ, and so obtain a knowledge of the scale of structure of the æther (if any).

Examples and Discussion of the "Normal State" in continuous media.

16. Before passing to further developments of the theory, I have thought it permissible to illustrate the foregoing ideas and concepts by a few mechanical illustrations.

I. *A Stretched String.*

17. Consider a dynamical system of which the kinetic and potential energies are respectively given by

$$2T = m(\dot{x}_1{}^2 + \dot{x}_2{}^2 + \ldots + \dot{x}_n{}^2), \quad \ldots \ldots \quad (10)$$

$$2V = \mu\{(x_0 - x_1)^2 + (x_1 - x_2)^2 + \ldots + (x_n - x_{n+1})^2\}, \quad (11)$$

and let n be very great.

If $x_0, x_1, x_2, \ldots x_n, x_{n+1}$ are coordinates of particles constrained to remain always in the same straight line, the system may be supposed to consist of a series of $n+2$ collinear particles, each attracting (or repelling) its neighbour according to the law of the direct distance, the two end particles being fixed in position. With a slight change in the meaning of the symbols, the system may be supposed to consist of heavy particles connected by elastic strings. In the limit when n is made infinite, the system will represent a continuous one-dimensional elastic medium, or a stretched string capable of performing longitudinal vibrations only.

Regarding the system as a collection of particles, the properties of the " normal state " can be seen at once from an examination of the energy-function. The velocities $\dot{x}_1, \dot{x}_2, \ldots \dot{x}_n$ will be distributed according to Maxwell's law

$$Ae^{-hmu^2}du,$$

and, when n is infinite, the same can be shown to be true of the differences $x_0 - x_1, x_1 - x_2,$ &c. Thus in the normal state,

each particle has, on the average, kinetic energy $\frac{1}{2}RT$, and each element of stretched string has, on the average, potential energy $\frac{1}{2}RT$. In the normal state, these kinetic and potential energies are distributed at random, and without correlation, about the common mean-value $\frac{1}{2}RT$.

In the present investigation, however, we are concerned with expressing the energy of the normal state in terms of the energy of trains of waves.

It is readily found that the free vibrations, subject to $x_0 = 0$, $x_{n+1} = 0$, are given by

$$x_s = \sin \frac{sq\pi}{n+1} \quad (q=1, 2, \ldots n), \quad . \quad . \quad . \quad (12)$$

and the frequency of this vibration (p) is given by

$$p = 2\sqrt{\frac{\mu}{m}} \sin \frac{q\pi}{2(n+1)}. \quad . \quad . \quad . \quad (13)$$

If $\xi_1, \xi_2, \ldots \xi_n$ are the various principal coordinates, we may, from equation (12), write

$$x_s = \sum_{q=1}^{q=n} \xi_q \sin \frac{sq\pi}{n+1}. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Expressed in terms of the principal coordinates, equations (10) and (11) become

$$2T = \frac{1}{2} m(n+1)(\dot{\xi}_1{}^2 + \dot{\xi}_2{}^2 + \ldots + \dot{\xi}_n{}^2),$$

$$2V = 2\mu(n+1)\left(\xi_1{}^2 \sin^2 \frac{\pi}{2(n+1)} + \xi_2{}^2 \sin^2 \frac{2\pi}{2(n+1)} + \ldots + \xi_n{}^2 \sin^2 \frac{n\pi}{2(n+1)} \right).$$

We can now express the energy of the " normal state " in terms of the energies of free vibrations, or of trains of waves. Since each term in T and V has an average amount of energy $\frac{1}{2}RT$, it follows that each free vibration has average energy RT. From equation (12) it follows that the wave-length λ of the qth free vibration is $\frac{2}{q} \times$ (length of system). Hence the number of free vibrations for which λ lies between λ and $\lambda + d\lambda$ is $2\lambda^{-2}d\lambda$ per unit length of the system. The energy per unit length, of wave-length intermediate between λ and $\lambda + d\lambda$, is accordingly

$$2RT\lambda^{-2}d\lambda. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

This is the one-dimensional analogue of formula (9). It represents the energy of the random distribution of kinetic

and potential energies which we have already seen to obtain in the normal state, analysed into the energy of regular trains of waves.

We return to the discussion of this system in § 23.

II. *A Tube of Air.*

18. The next system to be considered will consist of the air inside a tube of uniform cross-section, closed at both ends. For simplicity, the molecules of air will be supposed to be similar and infinitely small spheres : let them be N in number.

The normal state of this system is one with which the Kinetic Theory of Gases has made us very familiar. As regards position, the molecules are distributed absolutely at random throughout the tube : as regards motion, each velocity-component is distributed according to Maxwell's law.

Let us consider the arrangement of positions first. Let us imagine the tube divided into a great number n of cells, each being of the same cross-section as the tube, and of volume ω.

An arrangement by which the molecules are placed at exactly equal distances apart, in some regular geometrical order, is of course a possible arrangement, but is no more typical of the normal state than would be a motion in which each molecule had exactly the same velocity. So also an arrangement in which each of the n cells into which the tube is divided contained exactly the same number N/n of molecules, is possible, but is not typical of the normal state. In the normal state, the numbers of molecules in the different cells will be distributed around the mean value N/n, according to a law which can be determined.

Consider a single arrangement in which the numbers of molecules in the n cells taken in order are $a_1, a_2, \ldots a_n$. In the limit, when n is made infinite, a knowledge of the values of $a_1, a_2, \ldots a_n$ will be equivalent to a knowledge of the density of the gas at every point of the tube. On expressing this, by Fourier's theorem, as a series of circular functions, we can represent the deviations from uniform density as due to the superposition of trains of waves.

Employing the conception of probability in the exact sense in which I have defined it elsewhere [*], the probability that an

---

[*] Phil. Mag. [6] v. p. 597, or 'Dynamical Theory of Gases,' p. 53.

arrangement selected at random shall have $a_1, a_2, \ldots a_n$ as the numbers of molecules in the successive n cells, is *

$$\frac{N!\; n^{-N}}{a_1!\; a_2!\; \ldots a_n!}.$$

Hence the probability that $\frac{a_1}{N}, \frac{a_2}{N} \ldots$ shall lie within limits $d\left(\frac{a_1}{N}\right), d\left(\frac{a_2}{N}\right) \ldots$, is

$$\frac{N!\; n^{-N} N^n}{a_1!\; a_2!\; \ldots a_n!} d\left(\frac{a_1}{N}\right) d\left(\frac{a_2}{N}\right). \quad . \quad . \quad . \quad (16)$$

Let us now transform variables from $a_1, a_2 \ldots a_n$ to $\xi_1, \xi_2, \ldots \xi_n$, where $\xi_1, \xi_2, \ldots \xi_n$ are given by the equations

$$\frac{a_s}{\omega} = \frac{N}{n\omega} + \sum_{q=1}^{q=n} \xi_q \sin\frac{qs\pi}{n}, \; \&c. \; (s=1, 2, \ldots n). \quad . \quad (17)$$

In the limit, when n is made infinite, this equation becomes

$$\nu - \nu_0 = \sum_{q=1}^{q=n} \xi_q \sin\frac{x}{l}q\pi,$$

where ν is the molecular-density at a distance x along the tube (supposed of the total length l), and ν_0 is the average value of ν (*cf.* equation (14)). Thus $\xi_1, \xi_2, \ldots \xi_n$ ultimately become proportional to the amplitudes of waves of wave-lengths $2l, \frac{1}{2}(2l), \frac{1}{3}(2l), \&c.$

Let expression (16), transformed to variables $\xi_1, \xi_2, \ldots \xi_n$, be supposed to become

$$f(\xi_1, \xi_2, \ldots \xi_n)\, d\xi_1, d\xi_2, \ldots d\xi_n;$$

so that this expression will measure the probability that $\xi_1, \xi_2, \ldots \xi_n$ shall lie within limits $d\xi_1, d\xi_2, \ldots d\xi_n$. From equation (17) we have

$$\frac{\partial}{\partial \xi_q}\left(\frac{a_s}{N}\right) = \frac{\omega}{N} \sin sq\frac{\pi}{n};$$

so that

$$\frac{\partial\left(\frac{a_1}{N}, \frac{a_2}{N}, \ldots\right)}{\partial(\xi_1, \xi_2, \ldots)} = \omega^n N^{-n} \Delta,$$

where Δ is the determinant whose (s, q) term is $\sin(sq\pi/n)$, a pure number and a constant.

* 'Dynamical Theory of Gases,' p. 39.

Thus we have

$$f(\xi_1, \xi_2, \ldots \xi_n) = \frac{N!\; n^{-N} N^n}{a_1!\; a_2! \ldots a_n!} \cdot \frac{\partial\left(\frac{a_1}{N}, \frac{a_2}{N}, \ldots\right)}{\partial(\xi_1, \xi_2, \ldots)}$$

$$= \frac{N!\; n^{-N} \omega^n \Delta}{a_1!\; a_2! \ldots a_n!}.$$

Using Sterling's approximation for the values of $N!, a_1!, \&c.$ this gives

$$\log f(\xi_1, \xi_2 \ldots \xi_n) = C - \sum_{s=1}^{s=n} (a_s + \tfrac{1}{2}) \log \frac{na_s}{N}, \quad . \;(18)$$

where C denotes the constant

$$\frac{n}{2} \log n - \frac{n-1}{2} \log (2\pi N) + n \log \omega + \log \Delta.$$

Write equation (17) in the form

$$a_s = a_0 + S_s, \quad . \quad . \quad . \quad . \quad . \;(19)$$

where

$$a_0 = \frac{N}{n}, \; S_s = \omega \sum_{q=1}^{q=n} \xi_q \sin \frac{qs\pi}{n},$$

then, since S_s may be supposed small compared with a_0, we have

$$(a_s + \tfrac{1}{2}) \log \frac{na_s}{N} = (a_0 + \tfrac{1}{2} + S_s) \log \left(1 + \frac{S}{a_0}\right)$$

$$= -\frac{2a_0 - 1}{4a_0^2} S_s^2,$$

as far as terms of order $(S_s/a_0)^2$. On summing, we obtain

$$\sum_{s=1}^{s=n} (a_s + \tfrac{1}{2}) \log \frac{na_s}{N} = -\frac{2a_0 - 1}{4a_0^2} \sum_{s=1}^{s=n} S_s^2$$

$$= -\frac{2a_0 - 1}{4a_0^2} \frac{n\omega^2}{2} (\xi_1^2 + \xi_2^2 + \ldots \xi_n^2).$$

Replacing $\dfrac{2a_0 - 1}{4a_0^2}$ by $\dfrac{1}{2a_0}$ in this, equation (18) becomes

$$\log f(\xi_1, \xi_2, \ldots \xi_n) = C - \kappa(\xi_1^2 + \xi_2^2 + \ldots \xi_n^2),$$

where

$$\kappa = \frac{n\omega^2}{4a_0} = \frac{\Omega}{4\nu_0},$$

and so we obtain as the law of distribution

$$f(\xi_1, \xi_2 \ldots \xi_n) d\xi_1 d\xi_2 \ldots d\xi_n = A e^{-\kappa(\xi_1^2 + \xi_2^2 + \ldots \xi_n^2)} d\xi_1 d\xi_2 \ldots d\xi_n$$

$$. \quad . \quad . \quad . \;(20)$$

where A is a constant. It follows at once that $\xi_1^2, \xi_2^2, \ldots$ each have the same mean value, given by

$$\overline{\xi_1^2} = \overline{\xi_2^2} = \ldots \frac{1}{2\kappa} = \frac{2\nu_0}{\Omega}. \qquad . \qquad . \quad (21)$$

The next step is to find the potential energy of the trains of waves $\xi_1, \xi_2, \ldots \xi_n$. If p is the equilibrium pressure, and σ the condensation at any point, the potential energy V is given by

$$V = \tfrac{1}{2} p \iiint \sigma^2 \, dx \, dy \, dz,$$

where V is measured from the equilibrium configuration, and all heat-energy is treated as kinetic. The condensation in in the sth cell, say σ_s, is, by equation (19),

$$\sigma_s = \frac{a_s - a_0}{a_0} = \frac{S_s}{a_0} = \frac{\omega}{a_0} \sum_{q=1}^{q=n} \xi_q \sin \frac{qs\pi}{n},$$

so that

$$V = \tfrac{1}{2} p \, \Sigma \, \sigma_s^2 \, \omega$$

$$= \frac{p\Omega}{4\nu_0^2} (\xi_1^2 + \xi_2^2 + \ldots \xi_n^2),$$

or, since $p = R\nu_0 T$,

$$V = \tfrac{1}{4} RT \frac{\Omega}{\nu_0} (\xi_1^2 + \xi_2^2 + \ldots \xi_n^2).$$

Hence, by equation (21), each term in V has average energy $\tfrac{1}{2} RT$.

19. We next consider the partition of kinetic energy.

Suppose that of the N molecules, a number N' have x-components of velocity which are intermediate between u and $u + du$. In the "normal state" there is no correlation between velocity and positional coordinates, so that the N' molecules will be distributed between the n cells according to the same laws as the N molecules in the analysis just completed (§ 18).

Of these N' molecules, let the numbers in the different cells be $b_1, b_2, \ldots b_n$, and let $\eta_1, \eta_2, \ldots \eta_n$ be given (*cf.* equations (17)), by

$$\frac{b_s}{\omega} = \frac{N'}{n\omega} + \sum_{q=1}^{q=n} \eta_q \sin \frac{qs\pi}{n} \ (s = 1, 2, \ldots n), \quad . \quad (22)$$

then the η's will be distributed according to the law (*cf.* equation (20)),

$$A' \, e^{-\kappa'(\eta_1^2 + \eta_2^2 + \ldots \eta_n^2)} \, d\eta_1 \, d\eta_2 \ldots d\eta_n, \quad . \quad (23)$$

where $\kappa' = \Omega^2/4N'$.

That part of the x-momentum in the sth cell which arises from the motion of molecules with x-velocities between u and $u+du$, is

$$b_s\,mu = mu\frac{N'}{n} + \sum_{q=1}^{q=n} mu\,\omega\,\eta_q \sin\frac{qs\pi}{n}$$

The total momentum in the sth cell, obtained by summation, is accordingly

$$M_s = \sum_{u=-\infty}^{u=+\infty} b_s\,mu = \sum_{q=1}^{q=n} \zeta_q \sin\frac{qs\pi}{n}, \quad . \quad . \quad . \quad (24)$$

where

$$\zeta_q = \sum_{u=-\infty}^{u=+\infty} mu\,\omega\,\eta_q. \quad . \quad . \quad . \quad . \quad (25)$$

Equation (24) analyses the total x-momentum into the momenta of trains of waves. The total kinetic energy of these waves is

$$\frac{1}{2}\sum_{s=1}^{s=n} \frac{nM_s^2}{Nm} = \frac{n^2}{4Nm}(\zeta_1^2 + \zeta_2^2 + \dots \zeta_n^2). \quad . \quad . \quad (26)$$

To find the law of distribution of the ζ's we return to equation (23). The law of distribution of $mu\,\omega\,\eta_1$, $mu\,\omega\,\eta_2$, ... may from this equation be expressed in the form

$$A'' e^{-\kappa''\{(mu\,\omega\,\eta_1)^2+(mu\,\omega\,\eta_2)^2+\dots\}}\, d(mu\,\omega\,\eta_1)\, d(mu\,\omega\,\eta_2)\dots,$$

where $\kappa'' = \Omega^2/4N'u^2\omega^2m^2$. It follows at once that the law of distribution of the quantities ζ_1, ζ_2, ... given by equation (25) is [*]

$$A''' e^{-\kappa'''(\zeta_1^2+\zeta_2^2+\dots)}\, d\zeta_1\, d\zeta_2 \dots, \quad . \quad . \quad (27)$$

where

$$\frac{1}{\kappa'''} = \sum_{u=-\infty}^{u=+\infty} \frac{1}{\kappa''} = \frac{4\omega^2m^2}{\Omega^2}\sum_{u=-\infty}^{u=+\infty} N'u^2 = \frac{4mNRT}{n^2}.$$

---

[*] If m_1u_1, m_1v_1, m_1w_1 are distributed according to the law

$$C\,e^{-\frac{h}{m_1}\{(m_1u_1)^2+(m_1v_1)^2+(m_1w_1)^2\}}\, d(m_1u_1)\, d(m_1v_1)\, d(m_1w_1),$$

and m_2u_2, ... according to the law

$$C'\,e^{-\frac{h}{m_2}\{(m_2u_2)^2+\dots\}}\, d(m_2u_2)\dots,$$

then $m_1u_1+m_2u_2+\dots$, ... are distributed according to the law

$$C''e^{-\frac{h}{M}\{(m_1u_1+m_2u_2+\dots)^2+\dots\}}\, d(m_1u_1+m_2u_2+\dots)\dots,$$

where $M=m_1+m_2+\dots$. This result is obvious from physical considerations, or may of course be obtained by algebraic transformation.

Thus ζ_1, ζ_2, ... are distributed according to the law of trial and error, and the mean values of their squares are given by

$$\overline{\zeta_1{}^2} = \overline{\zeta_2{}^2} = \ldots = \frac{1}{2\kappa'''} = \frac{2m\mathrm{NRT}}{n^2}. \quad . \quad . \quad . \quad (28)$$

The energy of the trains of waves being, as has been seen in equation (26),

$$\frac{n^2}{4\mathrm{N}m}(\zeta_1{}^2 + \zeta_2{}^2 + \ldots + \zeta_n{}^2),$$

it now follows that the average value of each term is $\frac{1}{2}\mathrm{RT}$.

20. Thus we have seen that in the normal state the random motions and positions of the molecules result in certain departures from uniformity, both of density and of momentum. We have seen how these departures from uniformity can be regarded as due to regular trains of waves, and have effected their analysis into such trains of waves.

In any general motion of the medium, the kinetic and potential energies T V are given by

$$2\mathrm{T} = \alpha_1\dot{\phi}_1{}^2 + \alpha_2\dot{\phi}_2 + \ldots$$
$$2\mathrm{V} = \beta_1\phi_1{}^2 + \beta_2\phi_2{}^2 + \ldots,$$

where ϕ_1, ϕ_2 ... measure the amplitudes of different trains of waves. We have found that in the normal state the average value of each of the terms

$$\tfrac{1}{2}\alpha_1\dot{\phi}_1{}^2, \ \tfrac{1}{2}\alpha_2\dot{\phi}_2{}^2, \ \ldots \ \tfrac{1}{2}\beta_1\phi_1{}^2, \ldots$$

is the same, namely $\frac{1}{2}\mathrm{RT}$, the value given by the theorem of equipartition of energy. Incidentally, we have also verified that the values of $\dot{\phi}_1$, $\dot{\phi}_2$, ... ϕ_1, ... are ranged round their mean values according to the law of trial and error, as they ought to be.

To put the matter in another way, we have found that the law of distribution

$$\mathrm{A} \, e^{-hm(u_1{}^2 + v_1{}^2 + w_1{}^2 + u_2{}^2 + \cdots)} du_1 \, dv_1 \, dw_1 \, du_2 \ldots dx_1 dy_1 \, dz_1 \, dx_2 \ldots$$

is identical with, and may be transformed into, the law of distribution

$$\mathrm{A}' \, e^{-hm(\alpha_1\dot{\phi}_1{}^2 + \alpha_2\dot{\phi}_2{}^2 + \cdots + \beta_1\phi_1{}^2 + \beta_2\phi_2{}^2 + \cdots)} \, d\dot{\phi}_1 \, d\dot{\phi}_2 \ldots d\phi_1 \ldots.$$

The former law regards the energy as that of a system of

moving molecules: the latter regards the energy as that of a vibrating medium. Each law of distribution is that of the normal state. From the second, it follows at once (*cf.* equation (15)) that the law of partition of energy between vibrations of different wave-lengths is

$$2RT\lambda^{-2}\, d\lambda,$$

and it is easily seen that in the three-dimensional problem, the corresponding law is

$$4\pi RT\lambda^{-4}\, d\lambda.$$

These formulæ are of course true only for waves of length great compared with molecular distances. They require modification as we approach wave-lengths comparable with molecular distances.

III. *A Mechanical Model of the Æther.*

21. There appears to be no reason why the energy of the electromagnetic field cannot be treated similarly to that of a gas, except for the simplification that, so far as we know, no limitations need be introduced by the coarsegrainedness of the structure of the medium.

For simplicity let us consider a rectangular enclosure, and imagine it divided into equal cubical cells, each of edge l and volume ω. Let these cells be denoted by the numbers

$$000,\ 001,\ 002,\ \ldots\ 010,\ 011,\ \ldots$$

the cell pqr having as its Cartesian coordinates relative to the cell 000,

$$x = pl,\quad y = ql,\quad z = rl.$$

With any cell pqr we associate three coordinates ξ_{pqr}, η_{pqr}, ζ_{pqr}, and the three corresponding velocities $\dot\xi_{pqr}$, $\dot\eta_{pqr}$, $\dot\zeta_{pqr}$. We examine the motion of the system of which the energy function is given by

$$E = \frac{\omega}{8\pi} \underset{p}{\Sigma}\underset{q}{\Sigma}\underset{r}{\Sigma} \left\{ \dot\xi_{pqr}^2 + \dot\eta_{pqr}^2 + \dot\zeta_{pqr}^2 \right.$$
$$\left. + \frac{C^2}{l^2}\left[(\zeta_{p,q+1,r} - \zeta_{p,q,r} - \eta_{p,q,r+1} + \eta_{p,q,r})^2 + (\ldots)^2 + (\ldots)^2 \right] \right\}.$$
$$\cdots\ \ (29)$$

This function E is of course identical with the Hamiltonian function L′ of § 8, so that the equations of motion of the system can be written down at once (*cf.* equations (5)).

If we introduce new quantities α, β, γ, X, Y, Z, associated with each cell and given by

$$\alpha_{pqr} = \dot{\xi}_{pqr}$$
$$\left.\text{X}_{pqr} = \frac{\text{C}}{l} \left\{ \zeta_{p,\,q+1,\,r} - \zeta_{p,\,q,\,r} - \eta_{p,\,q,\,r+1} + \eta_{p,\,q,} \right\}, \right. \quad . \quad . \quad (30)$$

&c., then the energy-function becomes

$$\text{E} = \frac{\omega}{8\pi} \underset{p\ q\ r}{\Sigma\Sigma\Sigma} (\alpha^2 + \beta^2 + \gamma^2 + \text{X}^2 + \text{Y}^2 + \text{Z}^2). \quad . \quad (31)$$

If we make l and ω vanish in the limit, the value of E becomes

$$\text{E} = \frac{1}{8\pi} \iiint (\alpha^2 + \beta^2 + \gamma^2 + \text{X}^2 + \text{Y}^2 + \text{Z}^2)\, dx\, dy\, dz,$$

which is identical with the electromagnetic energy in free æther. The equations of motion obtained from equation (29) reduce, when l is made to vanish, to the equations

$$\frac{1}{\text{C}} \frac{d\text{X}}{dt} = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}$$

$$\frac{1}{\text{C}} \frac{d\alpha}{dt} = \frac{\partial \text{Z}}{\partial y} - \frac{\partial \text{Y}}{\partial z},$$

which are the electromagnetic equations in free æther.

Thus the mechanical system now under discussion becomes identical dynamically with the electromagnetic field when $l=0$. Any dynamical property of the present system which is independent of l must accordingly be a property of the electromagnetic field.

The energy function (29) is the three-dimensional analogue of the one-dimensional energy-function given by equations (10) and (11). It can accordingly be expressed as the energy of trains of waves, following the method of § 17. From this it follows that in the " normal state " the law of partition of energy between waves of different wave-lengths must be

$$8\pi\, \text{RT} \lambda^{-4} d\lambda, \quad . \quad . \quad . \quad . \quad (32)$$

so long as λ is large in comparison with l. Hence this must be the law of partition of energy in the " normal state " in an electromagnetic field, at any rate for waves which are long in comparison with the scale of structure (if any) of the æther.

S 2

The Rate of Approach to the "Normal State."

22. The " normal state " may be thought of as a sort of composite photograph of all possible states. Any features common to all states (or to all except an infinitesimal fraction of the whole), must be features of the normal state, and conversely. From this it follows (§ 10) that if there are any properties which a system tends to acquire, independently of the particular state from which the system starts, then these must be properties of the normal state. But it has not been proved, and cannot be proved, that a system will, in every case, tend to pass into the "normal state."

To take a well-known instance, the " normal state " of a gas inside a rectangular vessel is given by Maxwell's law, but if the system is started in such a way that the molecules all move on parallel paths perpendicular to one pair of faces, the system will not pass into the normal state at all.

23. Again, the energy of a non-dissipative medium, or conservative dynamical system capable of executing isochronous vibrations, can be expressed in the form

$$\left. \begin{array}{l} 2\mathrm{T} = \alpha_1 \dot{\phi}_1{}^2 + \alpha_2 \dot{\phi}_2{}^2 + \ldots \\ 2\mathrm{V} = \beta_1 \phi_1{}^2 + \beta_2 \phi_2{}^2 + \ldots \end{array} \right\}, \quad \cdot \quad \cdot \quad \cdot \quad (33)$$

where ϕ_1, ϕ_2, ... are the coordinates of the separate free vibrations. In the " normal state," we have

$$\tfrac{1}{2}\overline{\alpha_1 \dot{\phi}_1{}^2 + \beta_1 \phi_1{}^2} = \tfrac{1}{2}\overline{\alpha_2 \dot{\phi}_2{}^2 + \beta_2 \phi_2{}^2} = \ldots = \mathrm{RT}.$$

But in any free motion of the system, the quantities

$$\tfrac{1}{2}(\alpha_1 \dot{\phi}_1{}^2 + \beta_1 \phi_1{}^2), \ \tfrac{1}{2}(\alpha_2 \dot{\phi}_2{}^2 + \beta_2 \phi_2{}^2), \&\mathrm{c.}, \ \cdot \quad \cdot \quad (34)$$

retain through all time exactly those values with which they started. There is no tendency towards equalization of the values of these quantities, and therefore no tendency for the system to pass into the normal state.

For instance, in the system of § 17 (a string of particles) the motion consists of the propagation of trains of waves, without change of type or interchange of energy. As regards the practical problem of finding the final partition of energy, the existence of the " normal state " is of no account at all : the whole problem turns on the initial state of the system.

As regards the system of §§ 18–20 (the tube of air), we know that as a matter of fact the system does tend to assume the " normal state," which, as we have seen, is a state of random motion of the gas-molecules. The reason why this case differs from the last is that it is not permissible to

express the energy in the form (33), while at the same time treating the system as non-dissipative. If the energy is expressed in the form (33), the quantities (34) do not remain constant, for the energy they represent is subject to dissipation by viscosity. If we wish to treat the system as non-dissipative, we must regard it as an aggregation of molecules, and the energy can no longer be expressed in the form (33).

24. We come now to the electromagnetic system discussed in § 21. There seems to be no room for doubt that formula (32) accurately represents the partition of energy in the normal state, but the question of whether the system tends to pass into the normal state remains as yet unanswered.

Light reaches us from stars of which the parallax is too small to measure. The most refined measurements have never yet led us to suppose that its velocity depends either on its intensity or frequency. This and all other available evidence points to the fact that vibrations in free æther are isochronous and free from dissipation, or at least that they may be treated as such in the present investigation*. The energy may accordingly be expressed in the form (33), and the system treated as non-dissipative. It follows at once that the quantities (34) retain their original values—if not for ever, at least for a time incomparably greater than any that could be realized experimentally. It follows that a system consisting solely of free æther could never attain the normal state.

As soon, however, as matter is introduced into the enclosure, the problem assumes a different aspect. From the necessary interaction between matter and æther, it follows that the energy of the æther can no longer be expressed accurately in the form (33). The energies of the different vibrations into the æther no longer remain constant, for the matter supplies a means of interchange of energy between them. The question which now becomes of preponderating importance is that of the rate of transfer of energy.

25. It is known† that the rate at which energy is transferred to a vibration of frequency p is proportional to a

---

* A short calculation will show how safely we may neglect dissipation. Light reaches us from Arcturus, distant 2×10^{14} miles, and we have no reason to suppose that it is greatly dimmed on its way. However, as we can afford to be liberal in the allowance we make for dissipation, let us suppose that the light, by the time it reaches us, is dimmed to one-billionth (10^{-12}) of its original brightness. This means that the light has to travel 70,000 miles (a much greater distance than it could possibly be made to travel in any terrestrial experiment) before its energy is diminished even by one-millionth of one per cent.

† 'Dynamical Theory of Gases,' chap. ix.

factor $e^{-2p\beta}$, where β is a positive quantity. If the origin of heat-radiation is to be found in the collisions between electrons and atoms of matter, then β will be comparable with the time of collision, say, 7×10^{-14} sec. at ordinary temperatures. For yellow light $p = 3 \times 10^{15}$, so that $e^{-2p\beta} = e^{-420}$. Thus, purely as a matter of calculation, and apart from any special hypotheses or assumptions, we find that the rate at which visible light would be emitted, as the result of heat-radiation, from matter at ordinary temperatures, must necessarily be very slow. Thus, although the final law of partition of radiant energy in a perfectly reflecting enclosure containing some matter would be that of the "normal state," given by formula (32), yet it would require centuries to reach this final state, and the smallest departure from perfection in the reflecting power of the walls would result in this final state being impossible of attainment, even if infinite time were available.

So far from it being possible to assume infinite time or perfect reflecting power under experimental conditions, we find that we must assume exactly the reverse when light of short wave-length is concerned. For light of sufficiently small wave-length, the densest matter must be as transparent as is the atmosphere for ordinary light. Even for light of the wave-length of Röntgen rays, the walls of the experimental enclosure must be regarded as practically transparent. Thus energy of short wave-length disappears entirely from the enclosure within, say, 10^{-8} seconds after its emission, while the emissions of centuries would have to accumulate before the "normal state" could be established.

Conclusion.

26. It now appears certain that the observed partition of energy which Planck's formula (4) attempts to represent cannot be that of the "normal state"—at least for short wave-lengths.

We are at once confronted with the question : Why is it that the partition of energy given by this formula is, to all appearance at least, that of a final state which the system tends to assume, independently of the state from which it started? For the existence of such a final state, different from the "normal state," would at first sight seem contrary to the theorem of § 10.

In answering this question it is necessary to emphasize the distinction between "free" and "forced" oscillations The "normal state" is one in which the oscillations are free;

the system neither gains nor loses energy, and is free from all external disturbance for an unlimited time. In the observed state there is both a loss and a gain of energy (the two being of course equal in amount), and the oscillations are " forced." Any dynamical system subject to a loss of energy will, after a sufficient time, reach a state from which all traces of the initial conditions have disappeared; its state will depend solely on the forces which act upon it from outside. There exists a final state of this kind, to which every system necessarily tends when acted on by definite external agencies, and this final state may be something entirely different from the " normal state ": it is such a final state that Planck's formula represents. We may speak of this state as the " steady state ": it is the state reached as soon as the influence of the initial conditions has been dissipated away.

27. The " normal state " can depend only on the æther, but the " steady state " will depend in addition on the material agencies which force the vibrations in the æther. Consequently, Planck's formula (4) contains more constants than the formula (3) of the " normal state."

There is no *a priori* reason why there should not be different " steady state " formulæ corresponding to different kinds of matter : the vibrations forced in the æther might reasonably be expected to depend on the nature of the forcing agency. In point of fact it is found that the specification of the " steady state " involves only the temperature, and not the nature or structure, of the matter by which the ætherial vibrations are forced. This is easily seen to be a necessary consequence of Kirchhoff's law[*]. Once the truth of this law is accepted it is readily seen that there can be only one " steady state " corresponding to a given temperature, so that this steady state must be the same for all kinds of matter. Thus the constants which enter into the steady-state formula must be quantities which are common to all kinds of matter.

In Planck's formula there are two constants, c and k. Of these k is either identical with the gas-constant R or is

[*] If the views of the present paper are sound, the structure commonly called "The Thermodynamics of Radiation " requires modification, and the usual theoretical proof of Kirchhoff's law with it. We then have to fall back on experimental investigation as to the truth of the law. The law seems undoubtedly to be true, whether the usual theoretical proof of it is valid or not.

Lorentz, on the basis of the electrical structure of matter, has verified the truth of Kirchhoff's law for radiation of long wave-length (*Konink. Akad. van Wetenschappen*, Amsterdam, 1903, p. 678).

some numerical multiple of it (*cf.* § 5 and § 29, *infra*). The constant c is, on inspection of the formula, found to be of the same physical dimensions as E^2/K, where E is an electric charge and K an inductive capacity. The only quantity of these physical dimensions which can be constructed out of quantities common to all matter is e^2/K_0, where e is the charge of the electron, and K_0 is the inductive capacity of free æther. Thus we conclude that the c of Planck's formula must be some numerical multiplier of e^2/K_0 *.

28. Although the "steady state" is the same for all kinds of matter which actually exist, yet it depends on the quantity e, and so would be different for substances having different values of e, if such could exist †.

Imagine for the moment matter for which e is very small— matter for which we can say that $e = 0$. For such matter the partition of energy in the æther in the steady state is obtained by putting $c = 0$ in Planck's formula. It is therefore given by

$$f(\lambda, T) = 8\pi k T \lambda^{-4}. \quad . \quad . \quad . \quad . \quad (35)$$

This same formula of course gives also the partition of energy for actual matter for large values of λ.

Planck's value of $f(\lambda, T)$ differs from this in that it falls off in value as we come to small values of λ. Let us examine the physical significance of this.

29. Planck's formula differs from formula (35) as soon as $\dfrac{e^2}{K_0 R T \lambda}$ becomes appreciable—*i. e.*, as soon as λ is so small as to be comparable with $\dfrac{e^2}{K_0 R T}$. A physical interpretation of this last expression is readily found.

The motion of two electrons, or the motion of an electron and an atom (or molecule) can be regarded as the super-position of two motions. There is first the motion of the centre of gravity of the two bodies (which is unaffected by a collision or encounter between them), and second, there is the motion of the two bodies relative to their centre of

---

* It can be shown, from a consideration of physical dimensions (see Proc. Roy. Soc. lxxvi. p. 545), that, subject to a certain assumption, the "steady state" formula must be of the type

$$f(\lambda, T) = \frac{e^2}{K_0 \lambda^5} \phi \left(\frac{e^2}{K_0 R T \lambda} \right);$$

Planck's formula determines the form of the function ϕ.

† See Phil. Mag. [6] xii. p. 57.

gravity. The energy of the components of this last motion in any given direction is equal (on the average) to RT. Thus at the instant of closest approach in a collision or encounter, an amount of kinetic energy equal to RT must have been transformed into potential energy. If a is the distance of closest approach, the potential energy is $\dfrac{e^2}{K_0 a}$, so that

$$a = \frac{e^2}{K_0 RT}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (36)$$

Defining a by this equation, we see that Planck's formula differs from formula (35) as soon as λ becomes comparable with a, a being the average distance of closest approach in a collision.

When we consider matter in which, in the limit, we take $e = 0$, we have $a = 0$, and formula (35) gives the partition of energy in the issuing radiation. This is as it should be; collisions in which the electrons approach indefinitely close to one another will in this matter be of frequent occurrence, and radiation, even of very small wave-lengths, is produced at a rapid rate.

30. We now come to the final problem. Is formula (35), which gives the " steady state " for large values of λ, identical with formula (3), which gives the " normal state " ? Or, in other words, are k and R identical in § 5 ?

The work of Lorentz, already referred to (§§ 8, 27), provides incidentally an answer to this question. Lorentz finds, as the result of actual calculation, that what we have called the " steady state " formula, has for large values of λ the limiting form

$$8\pi RT\lambda^{-4}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (37)$$

and so establishes the identity of k and R. Lorentz's analysis, however, proceeds on certain definite assumptions, such as that the free electrons in a metal undergo instantaneous encounters, and that between these encounters they describe undisturbed free paths. It is therefore still necessary to examine whether, in order to compensate for the inaccuracy of these simplifying assumptions, it may not be necessary to correct Lorentz's expression by multiplication by a certain numerical factor [*]. If this is found to be necessary, then the identity of k and R will not have been proved.

---

[*] This is the only kind of correction which is possible without violating the physical dimensions of the formula in question.

31. Apart from this, the identity of k and R can, I think, be established by an argument of a very general nature, which does not depend on these special assumptions.

The continuous spectrum emitted by a solid must be granted to proceed from the motions of free electrons. A calculation, based on our knowledge of the total kinetic energy of these electrons and of the rate at which they radiate energy, will show that their rate of radiation of energy is very slow, when the time-scale is taken to be the average interval between two collisions. It follows that the kinetic energies of these electrons must be distributed in accordance with Maxwell's law about the mean value $\frac{3}{2}$RT. This theoretical result can be obtained independently of any assumptions as to the nature of free-paths, collisions, or forces by which the electrons are acted upon. Its truth has recently been confirmed in a very striking manner by the experiments of Richardson and Brown [*].

It follows that the motion of the electrons can be analysed into the motion of trains of waves by the analysis of § 19. Each of the trains of waves, into which the motion of the electrons can be analysed, will have kinetic energy appropriate to the temperature T. There will obviously be extremely rapid transfer of energy between these waves of electrons and the æther in which they are imbedded. Thus the æther in the interior of matter or in a cavity made in the matter, will immediately take up its equilibrium partition of energy appropriate to temperature T, namely, that given by the formula

$$8\pi RT\lambda^{-4}d\lambda. \quad . \quad . \quad . \quad . \quad . \quad (38)$$

There is a limit to the applicability of this argument. The analysis of the electron-motion into regular trains of waves holds only for wave-lengths great compared with the distances apart of the nearest electrons. Thus formula (38) will hold only for values of λ which are great compared with a, where a is given by equation (36). This is exactly what is given by Planck's formula, if the k of his formula is identified with R.

Princeton, Oct. 8, 1908.

[*] Phil. Mag. xvi. p. 353.