5. *Theory of the Relativistic Trajectories in a Gravitational Field of Schwarzschild.*

By

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[With 20 Figures.]

(Contribution from the Tokyo Astronomical Observatory.
Received Feb. 24, 1931. Communicated by K. HIRAYAMA.)

CONTENTS.

INTRODUCTION.

The relativistic line element of an external gravitational field due to a point-mass or a spherical mass of uniform density at rest was deduced by Einstein[1] and later by Schwarzschild.[2] In this paper I propose to study the motion of a particle in a gravitational field defined by this line element. It is supposed that the effect of this particle on the surrounding field is negligibly small.[3] The trajectory of such a particle is known to be a geodesic line in the four dimensional manifold

1) A. Einstein, Berl. Ber. (1915) 831; (1914) 1061; (1916) 688.

2) K. Schwarzschild, Berl. Ber. (1916) 424; H. Weyl, Phys. ZS. **20** (1919) 31; Ann. d. Phys. **59** (1919) 185. The rigorous deduction is due to Schwarzschild.

3) In this case we say that the particle is *massless*.

corresponding to that field of gravitation[1]. This problem was treated by Einstein,[2] Schwarzschild,[3] Droste[4] and others.[5] The motion was considered to occur in an ellipse with rotating perihelion in the first approximation from the Newtonian mechanics. This circumstance constituted one of the famous crucial tests of Einstein's theory of gravitation, that is, the secular motion of the perihelionof Mercury. Droste[4] and Morton[6] have found that various other forms of the trajectories are also possible, but neither of their works can be considered to be complete.

The purpose of this paper is to investigate the same problem in full detail from a different point of view. We form Hamilton-Jacobi's partial differential equation of the problem and integrate it by the method of the separation of the variables as was stated by Charlier,[7] and then discuss the motion by introducing elliptic functions, and obtain the analytic expansions according to the various types of possible motion, and finally deduce the motion of the perihelion in a quite general form without specification of the eccentricity of the orbit in the Newtonian sense. Connections between various types of motion are also discussed and explicit formulae for computing the position of the moving particle are obtained in detail. As a special case the trajectory for a light ray is also obtained.

This problem is easier than Kowalewski's theory of a top,[8] but much more interesting than Poinsot's top motion and the geodesics on an ellipsoid.[9]

1) Schwarzschild's line element of the external field of a non-rotating massive sphere of uniform density at rest is not an approximation. Only restriction is that the cosmological term is put to zero. Cf. de Donder, *Applications de la Gravifique Einsteinienne.* (1930), and Haag, *Le Problème de Schwarzschild.* (1931). The actual star is neither rigorously spherical referred to the co-ordinate system at rest, nor of uniform constant density referred to this co-ordinate system. It is in this sense that this line element of Schwarzschild is said to be an approximation.

2) 3) loc. cit.

4) Droste, Proc. Acad. Amsterdam. **17** (1914) 998; **18** (1915) 760.

5) W. de Sitter, Proc. Acad. Amsterdam, **19** (1916) 367; M.N. **76** (1916) 699; Greenhill, Phil. Mag. **41** (1921) 143; Forsyth, Proc. Roy. Soc. London, **97** A (1920) 145; M.N. **82** (1921) 2; Levi-Civita, Atti Lincei. Rendiconti. $\mathbf{26}_1$ (1917) 381, 458, 519; $\mathbf{26}_2$ (1917) 307; $\mathbf{27}_1$ (1918) 3, 183; $\mathbf{27}_2$ (1918) 220, 240, 283, 343; $\mathbf{28}_1$ (1919) 3, 101; Palatini, Il Nuovo Cimento. [vi] **14** (1917) 12; [vii] **26** (1923) 5.

6) Morton, Phil. Mag. **42** (1921) 511; also Whittaker, *Treatise on the Analytical Dynamics of Particles and Rigid Bodies.* Third Edition. (1927) Chap. XV.

7) Charlier, *Mechanik des Himmels.* Bd. **1** (1901).

8) Kowalewski, Acta Math. **12** (1889).

9) Halphen, *Traité des Fonctions Elliptiques et de leurs Applications.* Tome **2** (1888).

We start in Chap. I with a variational principle deduced from the dynamical equivalence of geometrical problems to facilitate our procedure of obtaining the relativistic trajectories in the four dimensional space-time-manifold. This same principle enables us to compute the trajectories for light rays as a special case. In Chap. II the Hamilton-Jacobi partial differential equation of the problem is formed and its characteristics are solved in order to compare our equations with those in the classical treatment. The integration of this Hamilton-Jacobi equation is undertaken in Chap. III by the method of the separation of the variables in the way of Stäckel. Then in Chap. IV the elliptic functions of Weierstrass are introduced to perform the complete integration of the problem. In Chap. V we discuss in detail the roots of the cubic, which is fundamental in the theory of elliptic functions, in comparison with the singularities appearing in the elliptic integrals in the solution. Various types of the distribution of these singularities among the roots of the cubic are counted up. The types of motion in the non-degenerate cases of the elliptic functions are obtained in Chap. VI with the explicit expansions for the motion. Those for the degenerate cases are treated in Chap. VII. The discussion of the results of Chap. VI and Chap. VII leads us to the knowledge how the various types of motion are correlated with each other and how they are transitioned to each other as we vary the constants of integration continuously. This is done in Chap. VIII. By the principle stated in Chap. I the trajectory of a light ray is easily obtained in Chap. IX as a special case of the motion of a particle. In Chap. X the nature of a quasi-elliptic motion is completely discussed. The form of the orbits, especially in the case very near to the Newtonian trajectories, is treated in full detail, with the generalisation of the formula of Einstein for the motion of the perihelion of the planets. The expansion is worked out in Chap. XI by the application of the Bessel functions of several variables.

Chapter I.

Variational Principle.

1. The trajectory of the motion of a massless particle in a gravitational field with a line element:

$$ds^2 = \sum_{\alpha,\beta}^{1,2,3,4} g_{\alpha\beta}\, dx_\alpha\, dx_\beta,$$

is, according to Einstein, a geodesic line in the four dimensional manifold x_i ($i=1, 2, 3, 4$), determined by the variational principle :

$$\delta\int ds = 0,$$

and its equations computed from this variational principle are

$$\frac{d^2x_i}{ds^2} + \sum_{\alpha,\beta}^{1,2,3,4} \left\{ \begin{matrix} \alpha\beta \\ i \end{matrix} \right\} \frac{dx_\alpha}{ds} \frac{dx_\beta}{ds} = 0. \quad (i=1, 2, 3, 4)$$

For the convenience of the later development of our theory I propose to replace this variational principle by an equivalent principle of a varied form deduced by the method of dynamical equivalence of the problems in geometry.

Consider a conservative holonomic dynamical system of four degrees of freedom. Let the time be expressed by σ and the kinetic energy[1] of this dynamical system be T, which is analytic in x_i and $\frac{dx_i}{d\sigma}$ ($i=1, 2, 3, 4$), and homogeneous and quadratic in $\frac{dx_i}{d\sigma}$. x_i's are the generalised co-ordinates. Suppose that the potential function[1] is zero. Then the Lagrangean function L and the Hamiltonian function H are both equal to T. By the variational principle of Hamilton the motion of this dynamical system is determined by[2]

$$\delta\int T d\sigma = 0. \tag{1}$$

The generalised momenta are defined by[3]

$$\frac{\partial T\left(x_i, \dfrac{dx_i}{d\sigma}\right)}{\partial\left(\dfrac{dx_j}{d\sigma}\right)} = p_j. \quad (j=1, 2, 3, 4) \tag{2}$$

1) I should like to make a remark at this point to avoid misunderstandings which may occur in the reader's mind. This dynamical meaning must not be confused with that in the relativistic trajectories. This dynamical system is rather an *image* of our problem of the motion of a particle in Schwarzschild's field. The terms, kinetic energy and potential function, are referred to that image, not to the motion of the particle in Schwarzschild's field. I introduce σ for the convenience of the treatment. It plays the rôle of an auxiliary variable to facilitate ourselves in carrying out our mode of thinking. σ is the time in this *image* dynamical system, while s is the arc length of a geodesic. In the usual correlation of dynamics and geometry the ratio of the differentials of these two variables determines the energy constant of the *image* dynamical system. *vide infra.*

2) Whittaker, *Analytical Dynamics.* Third Edition. (1927) 248.

3) Whittaker, *loc. cit.* 262.

In place of $\frac{dx_j}{d\sigma}$ $(j=1, 2, 3, 4)$ in T substitute their exressions in terms of p_j obtained by these equations and write the resulting expression $\Theta(x_i, p_i)$. As $H=T=L$, the variational principle (1) is equivalent to Hamilton-Jacobi's partial differential equation:

$$\frac{\partial W}{\partial \sigma}+\Theta\left(x_i, \frac{\partial W}{\partial x_i}\right)=0. \tag{3}$$

The characteristics of this equation are, by the well-known theory of a differential equation,[1]

$$\frac{dx_i}{\frac{\partial \Theta}{\partial p_i}}=\frac{-dp_j}{\frac{\partial \Theta}{\partial x_j}+\frac{\partial \Theta}{\partial W}p_j}=\frac{d\,W}{\sum_{k=1}^{4}\frac{\partial \Theta}{\partial p_k}\cdot p_k}=d\sigma, \quad (i, j, k=1, 2, 3, 4)$$

or, for our system (3),

$$\frac{dx_i}{d\sigma}=\frac{\partial \Theta}{\partial p_i}, \ \frac{dp_i}{d\sigma}=-\frac{\partial \Theta}{\partial x_i}. \quad (i=1, 2, 3, 4) \tag{4}$$

This is the Hamiltonian canonical system of differential equations.

2. Put

$$T=\frac{1}{2}\left(\frac{ds}{d\sigma}\right)^2, \tag{5}$$

with

$$ds^2=\sum_{\alpha, \beta}^{1, 2, 3, 4} g_{\alpha\beta}dx_\alpha dx_\beta, \tag{6}$$

where $g_{\alpha\beta}$'s are analytic functions of x_i. The Hamilton-Jacobi equation becomes in this case

$$\frac{\partial W}{\partial \sigma}+\frac{1}{2}\sum_{\alpha, \beta}^{1, 2, 3, 4} g^{\alpha\beta}\frac{\partial W}{\partial x_\alpha}\frac{\partial W}{\partial x_\beta}=0. \tag{7}$$

$g^{\alpha\beta}$ is contragradient to $g_{\alpha\beta}$. Here it is assumed that the determinant

1) Goursat, *Cours d'Analyse*, T. 2, 592; *Leçons sur l'Intégration des Equations aux Dérivées Partielles du Premier Ordre*. Chap. V; *Leçons sur le Problème de Pfaff*. Chap. VI; Forsyth, *Theory of Differential Equations*. Vol. 5, 146; G. Juvet, *Mécanique Analytique et Théorie des Quanta*. (1926); Frank u. Mises, *Differential- und Integralgleichungen der Mechanik und Physik*. Bd. 1 (1925) 518.

2) In this stage ds^2 is an abbreviation of the right hand member of (6). It has not the meaning of the line element square of the four dimensional Riemannian manifold· (5) is the kinetic energy of our image dynamical system.

formed by the matrix $g_{\alpha\beta}$ is not zero.[1] The characteristics of this equation are computed by (7) to be[2]

$$\frac{d^2x_i}{d\sigma^2}+\sum_{\alpha,\beta}^{1,2,3,4}\begin{Bmatrix}\alpha\ \beta\\ i\end{Bmatrix}\frac{dx_\alpha}{d\sigma}\frac{dx_\beta}{d\sigma}=0. \tag{8}$$

The same result will be obtained, if we start from (1), directly by substituting (5) and (6) in (1).

3. Now suppose that we have a constraint:[3]

$$s=C\sigma, \tag{9}$$

where C is a constant independent of x_i and of σ. Then (1) and (8) are formally transformed into

$$\delta\int ds=0, \tag{10}$$

$$\frac{d^2x_i}{ds^2}+\sum_{\alpha,\beta}^{1,2,3,4}\begin{Bmatrix}\alpha\ \beta\\ i\end{Bmatrix}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds}=0\,;\ (i=1,2,3,4) \tag{11}$$

and (10) and (11) are formally transformed into (1) and (8). This system is of the same form as Einstein's system of equations for the geodesics. If we prove that (11) is an actual consequence of (1) with the condition (9), then we shall be sure that our variational principle is equivalent to Einstein's principle (10) for the geodesics (11). Thus we shall be convinced of the legitimacy of our method of dynamical equivalence of the geometrical problem of determining the geodesics in the four dimensional Riemannian manifold with the line element (6).

4. The variational principle (1) under the restriction (9) is one of the conditioned variation. However it is easily shown that the problem can be treated as though there were no restriction at all and that we have only to put afterwards $s=C\sigma$ by determining this constant C so that

$$\sum_{\alpha,\beta}^{1,2,3,4} g_{\alpha\beta}\frac{dx_\alpha}{d\sigma}\frac{dx_\beta}{d\sigma}=C^2. \tag{12}$$

By the method of the Lagrangean multipliers in the calculus of variation,[4] we put

$$F=T+\lambda(2T-C^2)$$

1) The discussion will be undertaken at the end of Chap. II.

2) Those who find difficulties in deducing this equation are recommended to see: Backlund, Arkiv för Mat. Astr. och Fys. **14** (1918).

3) This special choice of the unit of time is equivalent to taking the energy constant of our image dynamical system equal to $\frac{1}{2}C^2$.

4) Cf., Bolza, *Vorlesungen über Variationsrechnug.* (1909) Kap. XI.

$$=(1+2\lambda)T-\lambda C^2,$$

where λ is the so-called Lagrangean multiplier and is a function of σ. Here

$$2T=\sum_{\alpha,\beta}^{1,2,3,4} g_{\alpha\beta}\frac{dx_\alpha}{d\sigma}\frac{dx_\beta}{d\sigma}, \tag{13}$$

and the variation takes the form:

$$\delta\int F d\sigma=0, \tag{14}$$

with an additional condition:

$$2T-C^2=0. \tag{15}$$

The Euler-Lagrange equations for this variation[1] are

$$(1+2\lambda)\frac{\partial T}{\partial x_i}-\frac{d}{d\sigma}\left\{(1+2\lambda)\frac{\partial T}{\partial\left(\frac{dx_i}{d\sigma}\right)}\right\}=0. \quad (i=1, 2, 3, 4) \tag{16}$$

This system admits an integral:[1]

$$F-\sum_{i=1}^{4}\frac{\partial F}{\partial\left(\frac{dx_i}{d\sigma}\right)}\cdot\left(\frac{dx_i}{d\sigma}\right)=\text{constant}, \tag{17}$$

or

$$(1+2\lambda)T+\lambda C^2=\text{constant}. \tag{18}$$

But, by (15), we get

$$\lambda=\text{constant}.$$

Hence (16) is transformed to

$$\frac{\partial T}{\partial x_i}-\frac{d}{d\sigma}\left(\frac{\partial T}{\partial\left(\frac{dx_i}{d\sigma}\right)}\right)=0. \quad (i=1, 2, 3, 4) \tag{19}$$

Thus, in our case in which (9) holds, the system is equivalent to

$$\frac{\partial T}{\partial x_i}-\frac{d}{d\sigma}\left(\frac{\partial T}{\partial\left(\frac{dx_i}{d\sigma}\right)}\right)=0, \quad (i=1, 2, 3, 4) \tag{19}$$

with an additional condition:

$$2T-C^2=0.$$

(19) is Euler's equation for the variation (1). Hence the above conditioned variation is equivalent to

1) Bolza, *loc. cit.*

$$\delta\int T d\sigma = 0, \tag{1}$$

in which we determine C^2 by the relation:

$$\sum_{\alpha,\beta}^{1,2,3,4} g_{\alpha\beta}\frac{dx_\alpha}{d\sigma}\frac{dx_\beta}{d\sigma} = C^2, \tag{12}$$

after obtaining x_i's as functions of σ by this unconditioned variational principle.

5. Next we have to prove that such a choice is possible. In our variational principle (1), the usual method is to form the Euler equations[1]:

$$\frac{\partial T}{\partial x_i} - \frac{d}{d\sigma}\left(\frac{\partial T}{\partial\left(\frac{dx_i}{d\sigma}\right)}\right) = 0. \quad (i=1, 2, 3, 4) \tag{19}$$

This system of equations admits an integral:[1]

$$T - \sum_{i=1}^{4}\frac{\partial T}{\partial\left(\frac{dx_i}{d\sigma}\right)}\cdot\left(\frac{dx_i}{d\sigma}\right) = \text{constant}.$$

By our assumption T is homogeneous and quadratic in $\frac{dx_i}{d\sigma}$. Hence

$$\sum_{i=1}^{4}\frac{\partial T}{\partial\left(\frac{dx_i}{d\sigma}\right)}\cdot\left(\frac{dx_i}{d\sigma}\right) = 2T.$$

Hence

$$T = \text{Constant};$$

or, by (5) and (10),

$$\frac{1}{2}\sum_{\alpha,\beta}^{1,2,3,4} g_{\alpha\beta}\frac{dx_\alpha}{d\sigma}\frac{dx_\beta}{d\sigma} = \text{constant}.$$

If we take the constant on the right hand side equal to $\frac{1}{2}C^2$, then it is evident that the choice of (12) is possible.

Hence C^2 in (9) can be expressed by the constants of integration of the system of differential equations (19).

6. Thus our process is as follows:

We solve the variational problem expressed by the principle:

$$\delta\int T d\sigma = 0, \tag{1}$$

as though there were no restriction at all. The system of differential equations of this problem is (8). The integral of this system introduces eight constants of integration. Determine C^2 as a function of these

1) Bolza, *loc. cit.*

constants of integration so that

$$\sum_{\alpha,\beta}^{1,2,3,4} g_{\alpha\beta} \frac{dx_\alpha}{d\sigma} \frac{dx_\beta}{d\sigma} = C^2.$$

Replace σ by Cs with C expressed by those constants. Then the equations (8) are equivalent to Einstein's equations for the geodesics in the four dimensional Riemannian manifold with the line element:

$$ds^2 = \sum_{\alpha,\beta}^{1,2,3,4} g_{\alpha\beta}\, dx_\alpha dx_\beta \quad . \tag{6}$$

The numbers of the constants of integration are the same both for (8) and for (11). As the potential function is assumed to be zero, $\frac{1}{2}C^2$ is the energy constant of our image dynamical system. Hence we get the following Lemma:

Lemma. The variational principle:

$$\delta \int ds = 0,$$

$$ds^2 = \sum_{\alpha,\beta}^{1,2,3,4} g_{\alpha\beta}\, dx_\alpha\, dx_\beta\ ,$$

is equivalent to

$$\delta \int T d\sigma = 0,$$

$$T = \frac{1}{2} \sum_{\alpha,\beta}^{1,2,3,4} g_{\alpha\beta} \frac{dx_\alpha}{d\sigma} \frac{dx_\beta}{d\sigma},$$

with an additional condition that, after determining x_i as functions of σ by treating the latter variational principle as though there were no restriction at all, we should determine a constant C^2 by the relation

$$\sum_{\alpha,\beta}^{1,2,3,4} g_{\alpha\beta} \frac{dx_\alpha}{d\sigma} \frac{dx_\beta}{d\sigma} = C^2, \tag{12}$$

as a function of the constants of integration, and then put

$$s = C\sigma, \tag{9}$$

with this value of C.

Thus under this condition we obtain Einstein's system of equations for the geodesics (11) directly from (1). (11) represents the geodesic lines in the four dimensional Riemannian manifold with the line element (6). Hence the dynamical system considered in the above is the dynamical equivalent of this geometrical problem to find the geodesic lines in the Riemannian manifold (6). It may be said to be an *image* (*abgebildetc*) dynamical system of our problem of relativistic trajectories.

The problems of dynamics are often solved by its analogy with geometry.[1] But our method of procedure goes in the opposite, that is, we are to solve the geometrical problem of Einstein by the process of analytical dynamics. This may be called the *method of dynamical equivalence.*

7. The process can be simplified by assuming $C^2=1$ in (9), that is, by putting

$$s=\sigma\,. \tag{9a}$$

The energy constant of our image dynamical system is then $\frac{1}{2}$. The equation (12) takes the form:

$$\sum_{\alpha,\beta}^{1,2,3,4} g_{\alpha\beta}\frac{dx_\alpha}{d\sigma}\frac{dx_\beta}{d\sigma}=1. \tag{12a}$$

This relation determines one of the eight constants of integration of (8). In fact, C^2 is expressed as a function of the eight constants by (12). Hence by putting $C^2=1$, we make the number of the independent constants of integration decrease by one. This is equivalent to taking the unit of time σ of our image dynamical system so that the value of the energy constant is always $\frac{1}{2}$. Thus we get the following Corollary to our Lemma.

1) Darboux, *Leçons sur la Théorie Générale des Surfaces.* T. 2 (1915) Chap. VI.-VIII; Synge, Phil. Trans. A, 226 (1926) 31; Whittaker, *Analytical Dynamics.* 254, 419; Poincaré, Trans. Amer. Math. Soc. 6 (1905) 237; Birkhoff, *ibid.* 18 (1917) 199.

It may be worth mentioning that Cinila, Il Nouvo Cimento, [vii] 16 (1918) 105, has applied Fermat's principle to the same problem. See, also, De Donder, *La Gravifique Einsteinienne.* (1921); Levi-Civita, *Der Absolute Differentialkalkül* (1928) Kap. VIII; *Fondamenti di Meccanica Relativistica.* (1928) Cap. I. Ogura, Tôhoku Math. Journ. 22 (1922) 14, has applied Darboux's method to a similar problem for light rays. He used the variational principle:

$$\delta\int\sqrt{-\Psi+h}\sqrt{\sum_{\alpha,\beta}^{1,2,3,4} g_{\alpha\beta}\,dx_\alpha dx_\beta}=0,$$

where Ψ is the potential function which is zero in our case and h is the energy constant. This is nothing but the principle of the least action. Jacobi, *Vorlesungen über Dynamik.* (1866) 44; Whittaker, *loc. cit.* Chap. IX. I might have been able to avoid the introduction of the image dynamical system, if I had referred to the Hamilton-Jacobi method in the calculus of variation. But I could not find any thorough treatise on it. See, Kneser, *Lehrbuch der Variationsrechnung.* (1925) 146.

Corollary. The variational principle:

$$\delta\int ds=0,$$

$$ds^2=\sum_{\alpha,\,\beta}^{1,\,2,\,3,\,4} g_{\alpha\beta}\,dx_\alpha\,dx_\beta\ ,$$

is equivalent to the variational principle:

$$\delta\int Td\sigma=0,$$

$$T=\frac{1}{2}\sum_{\alpha,\,\beta}^{1,\,2,\,3,\,4} g_{\alpha\beta}\,\frac{dx_\alpha}{d\sigma}\,\frac{dx_\beta}{d\sigma},$$

with an additional condition that, after determining x_i as functions of σ by treating the latter variational principle as though there were no restriction at all, we should determine one of the constants of integration by

$$\sum_{\alpha,\,\beta}^{1,\,2,\,3,\,4} g_{\alpha\beta}\,\frac{dx_\alpha}{d\tau}\,\frac{dx_\beta}{d\sigma}=1,$$

and then put $s=\sigma$.

This form is more convenient for the application, for we can dispense with one of the integration constants from the outset.

8. Let us turn back to our problem of finding the relativistic trajectories of massless particles in Schwarzschild's field of gravitation. We have three co-ordinates in space and one for time t. By the above procedure these four are expressed in terms of the so-called *proper time* s. It does not hurt the generality of our problem to take the origin of s such that $s=o$ for $t=o$. One of the constants of integration is thus disposed of. If we apply the Corollary, *only six constants are left independent.* Hence this suits our problem of finding the relativistic trajectories. ✱ N.B.

When we apply the principle in the Lemma, the proof of this proposition about the number of integration constants ought to be set off in another way.

If the Hamiltonian function in a system of canonical system of differential equations does not contain the independent variable explicitly, then the order of the differential equations is reduced by two according to the theorem of Lie and Levi-Civita.[1] If we assume that Θ contains neither σ nor x_4, then the enunciated property can be proved.

1) Goursat, *Leçons sur l'Intégration des Équations aux Dérivées Partielles du Premier Ordre.* (1921) Chap. XI; Levi-Civita, *Lezioni di Meccanica Razionale.* Vol. 2. Parte 2 (1927) Cap. X; Whittaker, *Analytical Dynamics.* Third Ed. (1927) Chap. XII.

This assumption is satisfied in our problem of Schwarzschild's field.

The system of differential equations:

$$\frac{dx_i}{ds}=\frac{\partial\Theta}{\partial p_i},\quad \frac{dp_i}{ds}=-\frac{\partial\Theta}{\partial x_i},\quad (i=1, 2, 3, 4)$$

which is obtained from (4) by applying our Lemma and in which one of the integration constants is contained in Θ, admits an integral:

$$\Theta+h=0,$$

with a constant of integration h, as Θ does not contain s. Arrange this in the form:

$$\Lambda+p_4=0,\quad p_4=\frac{dx_4}{ds},$$

assuming that $\frac{\partial\Theta}{\partial p_4}\neq 0$. The Pfaffian associated with these canonical equations is

$$\begin{aligned}
&p_1dx_1+p_2dx_2+p_3dx_3+p_4dx_4-\Theta ds\\
&\quad=p_1dx_1+p_2dx_2+p_3dx_3+p_4dx_4+hds\\
&\quad=p_1dx_1+p_2dx_2+p_3dx_3+hds-\Lambda\,(p_1, p_2, p_3, x_1, x_2, x_3, h)\,dx_4,
\end{aligned}$$

and the transformed equations are

$$\begin{cases}\dfrac{dx_i}{dx_4}=\dfrac{\partial\Lambda}{\partial p_i},\quad \dfrac{dp_i}{dx_4}=-\dfrac{\partial\Lambda}{\partial x_i},\quad (i=1, 2, 3)\\[2ex] \dfrac{ds}{dx_4}=\dfrac{\partial\Lambda}{\partial h},\quad \dfrac{dh}{dx_4}=0.\end{cases}$$

One of the integration constants is h and another occurs in the integration of the first equation of the last pair. Consider the first three pairs of equations only. Then the latter constants of integration does not come into play. Even the former constant h can be expressed by the other six, for there exists an integral:

$$\Lambda+h'=0,$$

for the first three pairs of equations with another constant of integration h', because in our problem Λ does not contain x_4 explicitly. The constant h involved in Λ is determined by this integral as a function of the other six constants. Hence, if we are dealing only with the first three pairs of these equations, two of the eight constants of integration can be left out of account. Hence our proposition is proved.

9. The relativistic trajectory of a light ray is obtained by taking

$$ds^2=0$$

according to Einstein. Hence if we put

$$C^2=0$$

in (9) and (12), then our principle provides us with a method of computing the trajectory of a light ray as a special case of the trajectories of massless particles. In this respect our Lemma is more convenient than our Corollary.

Chapter II.

Characteristics.

1. Consider a point-mass of mass at rest (*Ruhemass*) m situated at the origin of the co-ordinate system, or a spherical mass of mass at rest m with its centre at the origin of the co-ordinate system and with permanently constant uniform density at rest measured in this co-ordinate system. The line element square of the static field external to such a mass was given by Schwarzschild:

$$ds^2=c_0^2\left(1-\frac{\alpha}{r}\right)dt^2-\frac{1}{1-\frac{\alpha}{r}}dr^2-r^2d\varphi^2-r^2\sin^2\varphi d\theta^2, \tag{20}$$

where r, φ, θ are the polar co-ordinates with the mass as their origin and supposed to be at rest and measured by a scale at rest at the origin, and c_0 is a constant equal to the velocity of light at an infinite distance from the origin, and finally[1] $\alpha=2m$. The co-ordinates r, θ, φ and t can be identified with the Newtonian, and especially t is called the *co-ordinate time* or the *universal time* in contrast with the *proper time* s. N. K.

In order to see that our variational principle gives the same result as the classical, we form the characteristics of the Hamilton-Jacobi equation and compare them with the classical expressions for the trajectory.

1) As is well-known this number m is $\frac{\kappa}{c_0^2}$ times of that ordinarily used to represent the mass, where κ is Gauss's constant modified by the choice of the units of the length and of the time at rest at the origin. α appears as one of the integration constants of the general solution of Einstein's equations for the gravitational field. $\alpha=2m$ is obtained by comparing with the Newtonian mechanics. Our whole discussion is based on this line element. Hence we have nothing to do with the motion inside the stars. Neither we have any right to speak about the motion through a resisting medium, nor we have any appeal to the motion of a particle of so large a mass as to affect the surrounding gravitational field.

Take σ as the independent variable in our image dynamical system. By the method of the dynamical equivalence of the geometrical problem of finding the geodesic lines exposed in the last Chapter, Hamilton-Jacobi's partial differential equation of our problem is,

$$\frac{\partial W}{\partial \sigma}+\frac{1}{2c_0^2\left(1-\frac{\alpha}{r}\right)}\left(\frac{\partial W}{\partial t}\right)^2-\frac{1}{2}\left(1-\frac{\alpha}{r}\right)\left(\frac{\partial W}{\partial r}\right)^2-\frac{1}{2r^2\sin^2\varphi}\left(\frac{\partial W}{\partial \theta}\right)^2 -\frac{1}{2r^2}\left(\frac{\partial W}{\partial \varphi}\right)^2=0. \tag{21}$$

The characteristics of this equation are

$$\left.\begin{aligned}
&\frac{dt}{d\sigma}=\frac{1}{c_0^2\left(1-\frac{\alpha}{r}\right)}\frac{\partial W}{\partial t}, && \frac{d}{d\sigma}\left(\frac{\partial W}{\partial t}\right)=0,\\
&\frac{dr}{d\sigma}=-\left(1-\frac{\alpha}{r}\right)\frac{\partial W}{\partial r}, && \frac{d}{d\sigma}\left(\frac{\partial W}{\partial r}\right)=\frac{\frac{\alpha}{r^2}}{2c_0^2\left(1-\frac{\alpha}{r}\right)^2}\left(\frac{\partial W}{\partial t}\right)^2+\frac{\alpha}{2r^2}\left(\frac{\partial W}{\partial r}\right)^2\\
& && \qquad -\frac{1}{r^3}\left(\frac{\partial W}{\partial \varphi}\right)^2-\frac{1}{r^3\sin^2\varphi}\left(\frac{\partial W}{\partial \theta}\right)^2,\\
&\frac{d\varphi}{d\sigma}=-\frac{1}{r^2}\frac{\partial W}{\partial \varphi}, && \frac{d}{d\sigma}\left(\frac{\partial W}{\partial \varphi}\right)=-\frac{\cos\varphi}{r^2\sin^3\varphi}\left(\frac{\partial W}{\partial \theta}\right)^2,\\
&\frac{d\theta}{d\sigma}=-\frac{1}{r^2\sin^2\varphi}\frac{\partial W}{\partial \theta}, && \frac{d}{d\sigma}\left(\frac{\partial W}{\partial \theta}\right)=0.
\end{aligned}\right\} \tag{22}$$

In this system of equations θ is one of the so- called *ignorable* or *cyclic* co-ordinates of Helmholz.[1] Further, if we take σ as the independent variable, then t is also an ignorable co-ordinate. Hence we put

$$W=-K\sigma+ft+g\theta+V(r,\varphi), \tag{23}$$

where K, f, g are constants. Then the Hamilton-Jacobi equation is simplified to be

$$\frac{\partial V}{\partial \sigma}+\frac{f^2}{2c_0^2\left(1-\frac{\alpha}{r}\right)}-\frac{g^2}{2r^2\sin^2\varphi}-\frac{1}{2}\left(1-\frac{\alpha}{r}\right)\left(\frac{\partial V}{\partial r}\right)^2-\frac{1}{2r^2}\left(\frac{\partial V}{\partial \varphi}\right)^2=0. \tag{24}$$

The characteristics of this equation are

$$\frac{dr}{d\sigma}=-\left(1-\frac{\alpha}{r}\right)\frac{\partial V}{\partial r}, \quad \frac{d}{d\sigma}\left(\frac{\partial V}{\partial r}\right)=\frac{\frac{\alpha}{r^2}f^2}{2c_0^2\left(1-\frac{\alpha}{r}\right)^2}-\frac{g^2}{r^3\sin^2\varphi}$$

1) Whittaker, *loc. cit.* 55.

$$\left.\begin{aligned} &\qquad\qquad\qquad\qquad +\frac{\alpha}{2r^2}\left(\frac{\partial V}{\partial r}\right)^2-\frac{1}{r^3}\left(\frac{\partial V}{\partial \varphi}\right)^2, \\ &\frac{d\varphi}{d\sigma}=-\frac{1}{r^2}\frac{\partial V}{\partial \varphi}, \qquad \frac{d}{d\sigma}\left(\frac{\partial V}{\partial \varphi}\right)=-\frac{\cos\varphi}{r^2\sin^3\varphi}g^2. \end{aligned}\right\} \tag{25}$$

In order to obtain V, we put $g=0$. This is equivalent to taking the plane of the orbit in the plane $\theta=$const., that is, to restricting ourselves to the discussion of the trajectory in the manifold r, φ. The solutions obtained in this way are particular solutions of multiplicity ∞^4 instead of ∞^6 of the general solutions, where we suppose that two of the integration constants have been disposed of in the manner stated at the end of Chap. I. Then (25) takes the form

$$\left.\begin{aligned} &r^2\frac{d\varphi}{d\sigma}=AC, \\ &\frac{d}{d\sigma}\left(\frac{\frac{dr}{d\sigma}}{1-\frac{\alpha}{r}}\right)=\frac{-\frac{\alpha}{r^2}f^2}{2c_0^2\left(1-\frac{\alpha}{r}\right)^2}-\frac{\alpha}{2r^2}\frac{\left(\frac{dr}{d\sigma}\right)^2}{\left(1-\frac{\alpha}{r}\right)^2}+\frac{1}{r^3}A^2C^2; \end{aligned}\right\} \tag{26}$$

also by (22)

$$\frac{dt}{d\sigma}=\frac{f}{c_0^2\left(1-\frac{\alpha}{r}\right)}, \tag{27}$$

where A is a constant. By (9), (20) (26) and (27) we get

$$\left(\frac{dr}{ds}\right)^2=-\left(1-\frac{\alpha}{r}\right)\left(1+\frac{A}{r^2}\right)+\frac{f^2}{c_0^2C^2}, \tag{28}$$

$$r^2\frac{d\varphi}{ds}=A, \tag{29}$$

where we have determined the integration constant in (28) by comparing it with (20) and (27), and further

$$\left(\frac{A}{r^2}\frac{dr}{d\varphi}\right)^2+\frac{A^2}{r^2}=\frac{f^2}{c_0^2C^2}-1+\frac{\alpha}{r}+\frac{\alpha}{r}\frac{A^2}{r^2}. \tag{30}$$

C^2 is expressed by the other constants of integration according to the principle shown in the Lemma. By taking $A=\infty$ in (29) Eddington[1] has got the equation for a light ray. (26) and (27) are the same equations as those in the ordinary relativity mechanics. If we put $\frac{f}{c_0C}=c$

1) Eddington, *The Mathematical Theory of Relativity.* (1923) 86; Einstein, Ann. d. Phys. 49 (1916) 769; Berl. Ber. (1915) 831; de Sitter, M. N. 76 (1916) 699; Proc. Acad. Amsterdam. 19 (1917) 367.

(const.), then exactly the same equations as Eddington's are obtained.

The present method fails if the determinant g formed by the matrix $g_{\alpha\beta}$ vanishes. From (20) we get

$$g = -c_0^2 r^4 \sin^2 \varphi.$$

For $\varphi = 0$ or π, g vanishes. But by a suitable choice of the reference plane we can get rid of this singularity. Also for $r=0$, g vanishes. Physically we are considering only $r>0$. Hence this singularity, too, can be avoided. Thus g does not change its sign in our physical problem.[1)]

Ordinarily the motion of the perihelion is deduced from (29) and (30) directly as was done by Einstein and Eddington; or by the method of the variation of the elements in the theory of perturbation under the action of the perturbative force which can be obtained by replacing σ in our equations (25) and (26) by its expression in t from (27) and comparing it with the ordinary Newtonian equation in the way as was pursued by de Sitter. Our procedure consists in applying the method[2)] of the separation of the variables in Hamilton-Jacobi's equation as stated by Charlier[3)] and in introducing elliptic functions so as to be able to discuss the trajectories in all possible cases and without specification of the form of the orbit and to deduce the analytical expressions for computing the position of the particle in any of those possible types of motion.

Chapter III.

Integration of Hamilton-Jacobi's Equation.

Let us return to our Hamilton-Jacobi's partial differential equation

1) Jacobi, *Vorlesungen über Dynamik.* (1866) 47; Bolza, *loc. cit.* 231, 241; Hadamard, Journ. de Math. [v] **3** (1897) 331. I do not see the proof for higher dimensions. But my argument lies in the fact that $g<0$ in the domain under our consideration. This is due to the principle of relativity. Cf., Hilbert, Göttingen Nachrichten. (1917) 53; Math. Ann. **92** (1924) 1.

2) Charlier, *Mechanik des Himmels.* Bd. **1** (1901).

3) Stäckel, Dissertation; O. Staude, Acta Math. **10** (1887) 18; **11** (1888) 303; Painlevé, *Leçons sur l'Intégration des Équations Différentielles de la Mécanique.* 201; C. Neumann, Crelle J. **54** (1859) 46; Weierstrass, Monatsberichte d. Berl. Akad. (1866) 97; O. Staude, Math. Ann. **29** (1887) 468; Crelle J. **105** (1888) 298; Acta Math. **11** (1888) 303; P. Stäckel, Crelle J. **107** (1891) 319; Math. Ann. **42** (1893) 537; Levi-Civita, Math. Ann. **59** (1904) 383; Stäckel, Crelle J. **128** (1905) 222; &c.

(21) and proceed by the method of the Corollary of Chap. I. Put

$$W=-K\sigma+V_1(t)+V_2(r)+V_3(\varphi)+V_4(\theta), \tag{31}$$

then, (21) takes the form

$$-K+\frac{1}{2c_0^2\left(1-\frac{\alpha}{r}\right)}\left(\frac{dV_1}{dt}\right)^2-\frac{1}{2}\left(1-\frac{\alpha}{r}\right)\left(\frac{dV_2}{dr}\right)^2$$
$$-\frac{1}{2r^2}\left(\frac{dV_3}{d\varphi}\right)^2-\frac{1}{2r^2\sin^2\varphi}\left(\frac{dV_4}{d\theta}\right)^2=0. \tag{32}$$

Form this equation we get successively

$$\left.\begin{aligned}
&r^2\left\{-K+\frac{1}{2c_0^2\left(1-\frac{\alpha}{r}\right)}\left(\frac{dV_1}{dt}\right)^2-\frac{1}{2}\left(1-\frac{\alpha}{r}\right)\left(\frac{dV_2}{dr}\right)^2\right\}\\
&\qquad=\frac{1}{2}\left(\frac{dV_3}{d\varphi}\right)^2+\frac{1}{2\sin^2\varphi}\left(\frac{dV_4}{d\theta}\right)^2=\frac{h_1^2}{2},\\
&\sin^2\varphi\left\{h_1^2-\left(\frac{dV_3}{d\varphi}\right)^2\right\}=\left(\frac{dV_4}{d\theta}\right)^2=h_2^2,\\
&\left(\frac{dV_1}{dt}\right)^2=2c_0^2\left\{\frac{h_1^2}{2r^2}+K+\frac{1}{2}\left(1-\frac{\alpha}{r}\right)\left(\frac{dV_2}{dr}\right)^2\right\}\left(1-\frac{\alpha}{r}\right)=h_3^2c_0^2,
\end{aligned}\right\} \tag{33}$$

where h_1, h_2, h_3 are constants. Then[1)]

$$\left.\begin{aligned}
&V_1=c_0\int h_3dt=c_0h_3t,\\
&V_2=\int\sqrt{\frac{2}{1-\frac{\alpha}{r}}\left\{1-\frac{h_1^2}{2r^2}+\frac{h_3^2}{2\left(1-\frac{\alpha}{r}\right)}-K\right\}}.\ dr,\\
&V_3=\int\sqrt{h_1^2-\frac{h_3^2}{\sin^2\varphi}}.\ d\varphi,\\
&V_4=-\int h_2d\theta=-h_2\theta.
\end{aligned}\right\} \tag{34}$$

Take a system of canonical constants β_0, β_1, β_2, β_3, conjugate to K, h_1, h_2, h_3. The required solution is, by the general rule of Jacobi's method,

$$\left.\begin{aligned}
&\frac{\partial W}{\partial K}=\beta_0, && y_0=\frac{\partial W}{\partial x_0},\\
&\frac{\partial W}{\partial h_i}=\beta_i, && y_i=\frac{\partial W}{\partial x_i},\ (i=1,\ 2,\ 3)
\end{aligned}\right\} \tag{35}$$

with an additional condition :

1) The minus sign before h_2 is taken in order to conform with the usual definition of the inclination I in (44).

$$1=c_0^2\left(1-\frac{\alpha}{r}\right)\left(\frac{dt}{d\sigma}\right)^2-\frac{1}{1-\frac{\alpha}{r}}\left(\frac{dr}{d\sigma}\right)^2-r^2\left(\frac{d\varphi}{d\sigma}\right)^2-r^2\sin^2\varphi\left(\frac{d\theta}{d\sigma}\right)^2, \tag{36}$$

by our Corollary. Substituting (31) (34) in (35) we have

$$\left.\begin{aligned}
-\beta_0&=\sigma+\int\frac{dr}{\sqrt{R(r)}},\\
\beta_1&=-\int\frac{h_1dr}{r^2\sqrt{R(r)}}+\int\frac{h_1}{\sqrt{\Phi(\varphi)}}d\varphi,\\
\beta_2&=-\int\frac{h_2d\varphi}{\sin^2\varphi\sqrt{\Phi(\varphi)}}-\theta,\\
\beta_3&=\int\frac{h_1rdr}{(r-\alpha)\sqrt{R(r)}}+c_0t,
\end{aligned}\right\} \tag{37}$$

where

$$\left.\begin{aligned}
R(r)&=(h_3^2-2K)+\frac{2\alpha K}{r}-\frac{h_1^2}{r^2}+\frac{\alpha h_1^2}{r^3},\\
\Phi(\varphi)&=h_1^2-\frac{h_2^2}{\sin^2\varphi}.
\end{aligned}\right\} \tag{38}$$

Let

$$u=\frac{\alpha}{r},$$

then

$$\frac{h_1dr}{\sqrt{R(r)}}=-\frac{\alpha^2du}{u^2\sqrt{U(u)}},$$

with

$$U(u)=\frac{\alpha^2(h_3^2-2K)}{h_1^2}+\frac{2K\alpha^2}{h_1^2}u-u^2+u^3. \tag{39}$$

Then (37) takes the form

$$\left.\begin{aligned}
\frac{h_1d\sigma}{\alpha^2}&=\frac{du}{u^2\sqrt{U(u)}},\\
\frac{h_1d\varphi}{\sqrt{\Phi(\varphi)}}&=-\frac{du}{\sqrt{U(u)}},\\
d\theta&=-\frac{h_2}{\sin^2\varphi}\frac{d\varphi}{\sqrt{\Phi(\varphi)}},\\
c_0dt&=\frac{\alpha^2h_3}{h_1}\frac{du}{(1-u)u^2\sqrt{U(u)}}.
\end{aligned}\right\} \tag{40}$$

(36) becomes

$$d\sigma^2=c_0^2(1-u)dt^2+\frac{\alpha^2}{u^2}d\varphi^2+\frac{\alpha^2}{u^2}\sin^2\varphi d\theta^2+\frac{1}{1-u}\frac{\alpha^2}{u^4}du^2=0. \tag{41}$$

Substituting (39) (40) in this equation we have

$$K=\frac{1}{2}. \tag{42}$$

This determines K so that (12) is satisfied. Thus σ is defined by

$$d\sigma^2=c_0^2\left(1-\frac{\alpha}{r}\right)dt^2-\frac{1}{1-\frac{\alpha}{r}}dr^2-r^2d\varphi^2-r^2\sin^2\varphi d\theta^2. \tag{43}$$

According to the principle of Chap. I, we can write s instead of σ by putting $K=\frac{1}{2}$. The constant of integration β_0 is superfluous for determining the relativistic trajectories of a particle in a gravitational field, as it does not hurt the generality of the result to fix at the outset such that s should be zero with t. Thus

$$U(u)=u^3-u^2+\frac{\alpha^2}{h_1^2}u+\frac{\alpha^2(h_3^2-1)}{h_1^2}. \tag{39a}$$

In this connection it is interesting to see the equation for the path of a light ray. For a light ray we ought to have $ds=0$. This will be fulfilled only if $K=0$. But this circumstance can not be realised in our problem. Hence the path of the moving particle in a relativistic gravitational field of Schwarzschild can nowhere at ordinary points coincide with that of a light ray.

Now introduce I through

$$\cos I=\frac{h_2}{h_1}, \qquad \text{or} \qquad \sin I=\sqrt{\frac{h_1^2-h_2^2}{h_1^2}}, \tag{44}$$

where I denotes the *inclination* of the orbital plane to the plane of reference. And also denote the *argument of latitude* by ψ. From the spherical triangle $\mathfrak{T}$ formed by the orbital plane, the reference plane and the meridian, we have the relation:

$$\cos\varphi=\sin I\sin\psi, \tag{45}$$

where φ stands for the *co-latitude*. Hence

$$\frac{d\varphi}{\sqrt{\Phi(\varphi)}}=-\frac{d\psi}{h_1}. \tag{46}$$

Hence the second equation of (40) takes the form:

$$\frac{du}{\sqrt{U(u)}}=d\psi. \tag{47}$$

The third equation of (40) is transformed, by using (45) and (46), into

$$d\theta=\frac{\cos I}{\cos^2\varphi}d\psi=\frac{\cos I d\psi}{1-\sin^2 I\sin^2\psi}=\frac{\cos I}{2}\left(\frac{d\psi}{1-\sin I\sin\psi}+\frac{d\psi}{1+\sin I\sin\psi}\right)$$

$$=d\left[\tan^{-1}\left(\frac{\tan\frac{\psi}{2}+\sin I}{\cos I}\right)+\tan^{-1}\left(\frac{\tan\frac{\psi}{2}-\sin I}{\cos I}\right)\right]$$

$$=d.\tan^{-1}(\tan\psi.\cos I),$$

that is,

$$\tan(\theta+\beta_2)=\tan\psi\cos I. \tag{48}$$

This is one of the fundamental formulae of the spherical rectangular triangle $\mathfrak{T}$. $\theta+\beta_2$ denotes the *longitude*. Thus it is seen that our orbital plane lies on the same fixed plane in the whole duration of the motion. Hence we are led to the following system of equations:

$$\left.\begin{aligned}\frac{h_1}{\alpha^2}ds&=\frac{du}{u^2\sqrt{U(u)}},\\ 0&=\frac{du}{\sqrt{U(u)}}-d\psi,\\ 0&=\frac{du}{(1-u)u^2\sqrt{U(u)}}-\frac{h_1c_0}{\alpha^2h_3}dt,\end{aligned}\right\} \tag{49}$$

with

$$U(u)=u^3-u^2+\frac{\alpha^2}{h_1^2}u+\frac{\alpha^2(h_3^2-1)}{h_1^2}. \tag{39a}$$

Capter IV.

Introduction of Elliptic Functions.

We introduce the elliptic functions of Weierstrass to solve (49), which is nothing but a simple case of the inversion of algebraic integrals. Denote by u_1, u_2, u_3 the three roots of the equation $U(u)=0$.

$$\left.\begin{aligned} &u_1+u_2+u_3=1, \quad u_2u_3+u_3u_1+u_1u_2=\frac{\alpha^2}{h_1^2}, \\ &u_1u_2u_3=-\frac{\alpha^2(h_3^2-1)}{h_1^2}. \end{aligned}\right\} \tag{50}$$

Put

$$x=u-\frac{1}{3}, \quad e_i=u_i-\frac{1}{3}, \quad (i=1, 2, 3) \tag{51}$$

then the second equation of (49) takes the form:

$$d\psi=\frac{dx}{\sqrt{(x-e_1)(x-e_2)(x-e_3)}}. \tag{52}$$

Introduce the $\wp$-function of Weierstrass with the half double periods ω and ω', where $\wp\omega=e_1$, $\wp(\omega+\omega')=e_2$, $\wp\omega'=e_3$ and $e_1+e_2+e_3=0$. Then (52) is integrated with the constants of integration in (37), in which K is determined by (42) and β_0 is superfluous, and we get

$$x=\wp\left(\frac{\psi+\beta_1}{2}\right), \tag{53}$$

and

$$u=\frac{\alpha}{r}=\wp\left(\frac{\psi+\beta_1}{2}\right)+\frac{1}{3}. \tag{54}$$

The first and the third equations of (40) become

$$\frac{h_1}{\alpha^2}ds=\frac{d\psi}{\left\{\wp\left(\frac{\psi+\beta_1}{2}\right)+\frac{1}{3}\right\}^2}=\frac{dx}{\left(x+\frac{1}{3}\right)^2\sqrt{(x-e_1)(x-e_2)(x-e_3)}}, \tag{55}$$

$$\begin{aligned} \frac{h_1c_0}{\alpha^2h_3}dt&=\frac{d\psi}{\left\{\frac{2}{3}-\wp\left(\frac{\psi+\beta_1}{2}\right)\right\}\left\{\wp\left(\frac{\psi+\beta_1}{2}\right)+\frac{1}{3}\right\}^2} \\ &=\frac{dx}{\left(\frac{2}{3}-x\right)\left(x+\frac{1}{3}\right)^2\sqrt{(x-e_1)(x-e_2)(x-e_3)}}. \end{aligned} \tag{56}$$

Now, if we differentiate an well-known formula of the elliptic functions:[1]

$$\zeta(x+y)-\zeta(x-y)-2\zeta y=-\frac{\wp'y}{\wp x-\wp y}, \tag{57}$$

then we get

$$\wp(x+y)+\wp(x-y)-2\wp y=\frac{\wp'^2y}{(\wp x-\wp y)^2}+\frac{\wp''y}{\wp x-\wp y}. \tag{58}$$

1) Halphen, *Traité des Fonctions Elliptiques et de leurs Applications*. Vol. 1 (1886) 205.

The integration of (55) and (56) can be performed by using these two formulae.[1)]

At first put

$$\wp y=-\frac{1}{3}. \tag{59}$$

Then the invariants of the elliptic functions g_2, g_3 can be written

$$g_2=-4(e_1e_2+e_2e_3+e_3e_1)=4\left(\frac{1}{3}-\frac{\alpha^2}{h_1^2}\right),$$

$$g_3=4e_1e_2e_3=4\left(\frac{2}{27}+\frac{2}{3}\frac{\alpha^2}{h_1^2}-\frac{\alpha^2h_3^2}{h_1^2}\right),$$

$$\Delta=g_2^3-27g_3^2,$$

and further we have

$$\wp'^2y=4(\wp y-e_1)(\wp y-e_2)(\wp y-e_3)=\frac{4\alpha^2(h_3^2-1)}{h_1^2}, \tag{60}$$

$$\wp''y=6\wp^2y-\frac{1}{2}g_2=\frac{2\alpha^2}{h_1^2}. \tag{61}$$

By (57), (58) and

$$\frac{d}{dx}\zeta x=-\wp x, \quad \frac{d}{dx}\log\sigma x=\zeta x,$$

we get

$$\wp'^2y\int\frac{dx}{(\wp x-\wp y)^2}=-\zeta(x+y)-\zeta(x-y)-2x\wp y$$
$$+\frac{\wp''y}{\wp'y}\left\{\log\frac{\sigma(x+y)}{\sigma(x-y)}-2x\zeta y\right\}. \tag{62}$$

Hence (55) can be integrated in the form :

$$\frac{h_1}{2\alpha^2}(s+\beta_0)=\frac{1}{\wp'^2y}\left[-\zeta\left(\frac{\psi+\beta_1}{2}+y\right)-\zeta\left(\frac{\psi+\beta_1}{2}-y\right)-(\psi+\beta_1)\wp y\right.$$
$$\left.+\frac{\wp''y}{\wp'y}\left\{\log\frac{\sigma\left(\frac{\psi+\beta_1}{2}+y\right)}{\sigma\left(\frac{\psi+\beta_1}{2}-y\right)}-(\psi+\beta_1)\zeta y\right\}\right]. \tag{63}$$

1) Evidently the integrated forms of (55) and (56) are elliptic integrals of the third kind. They can be transformed to the sums of elliptic integrals of the third kind and of the second kind in Legendre's normal form. The appearance of the elliptic integrals of the third kind necessitates the forthcoming of logarithmic functions.

To integrate (56) we resolve the right hand member into partial fractions and put

$$\wp z = \frac{2}{3}, \tag{64}$$

so that

$$\wp'^2 z = \frac{4\alpha^2 h_3^2}{h_1^2},$$

and then integrate. Thus we get

$$\frac{h_1}{\alpha^2 h_3}(c_0 t + \beta_3) = \frac{1}{\wp' z}\left\{(\psi+\beta_1)\zeta z - \log\frac{\sigma\left(\frac{\psi+\beta_1}{2}+z\right)}{\sigma\left(\frac{\psi+\beta_1}{2}-z\right)}\right\}$$

$$-\frac{1}{\wp' y}\left\{(\psi+\beta_1)\zeta y - \log\frac{\sigma\left(\frac{\psi+\beta_1}{2}+y\right)}{\sigma\left(\frac{\psi+\beta_1}{2}-y\right)}\right\}\cdot\left(1-\frac{\wp'' y}{\wp'^2 y}\right)$$

$$+\frac{1}{\wp'^2 y}\left\{\zeta\left(\frac{\psi+\beta_1}{2}+y\right)+\zeta\left(\frac{\psi+\beta_1}{2}-y\right)+(\psi+\beta_1)\wp y\right\}. \tag{65}$$

Now

$$\left.\begin{aligned} &\wp y = -\frac{1}{3}, \quad (\wp' y)^2 = \frac{4\alpha^2(h_3^2-1)}{h_1^2}, \quad \wp'' y = \frac{2\alpha^2}{h_1^2}, \\ &\wp z = \frac{2}{3}, \quad (\wp' z)^2 = \frac{4\alpha^2 h_3^2}{h_1^2}, \end{aligned}\right\} \tag{66}$$

and by the formulae:[1]

$$\left.\begin{aligned} &\wp w = -\frac{\eta}{\omega} + \left(\frac{\pi}{2\omega}\right)^2 \frac{1}{\sin^2\frac{\pi w}{2\omega}} - 2\left(\frac{\pi}{\omega}\right)^2 \sum_{n=1}^{\infty} \frac{n q^{2n}}{1-q^{2n}} \cos\frac{n\pi w}{\omega}, \\ &\zeta w = \frac{\eta w}{\omega} + \frac{\pi}{2\omega}\cot\frac{\pi w}{2\omega} + \frac{2\pi}{\omega}\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin\frac{n\pi w}{\omega}, \\ &\log\sigma w = \frac{\eta w^2}{2\omega} + \log\left(\frac{2\omega}{\pi}\sin\frac{\pi w}{2\omega}\right) + 2\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\,\frac{1-\cos\frac{n\pi w}{\omega}}{n}, \\ &\eta\omega = \frac{1}{12}\pi^2 - 2\pi^2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1+q^{2n})^2}, \\ &q = e^{-\frac{\pi\omega'}{\omega i}}. \end{aligned}\right\} \tag{67}$$

1) Halphen, *Traité des Fonctions Elliptiques et de leurs Applications.* Vol. 1. pp. 426, 428, 404.

(54), (63) and (65) can be expanded in the following form:

$$\frac{h_1}{2\alpha^2}(\wp' y)^2(s+\beta_0)=\left\{-\wp y-\frac{\eta}{\omega}+\frac{\wp'' y}{\wp' y}\left(\frac{\eta y}{\omega}-\zeta y\right)\right\}(\psi+\beta_1)$$

$$-\frac{\pi}{\omega}\,\frac{\sin\frac{\pi}{2\omega}(\psi+\beta_1)}{\cos\frac{\pi y}{\omega}-\cos\frac{\pi}{2\omega}(\psi+\beta_1)}+\frac{\wp'' y}{\wp' y}\log\frac{\sin\frac{\pi}{2\omega}\left(\frac{\psi+\beta_1}{2}+y\right)}{\sin\frac{\pi}{2\omega}\left(\frac{\psi+\beta_1}{2}-y\right)}$$

$$+\sum_{n=1}^{\infty}\left(-\frac{\pi}{\omega}\cos\frac{n\pi}{\omega}y+\frac{\wp'' y}{\wp' y}\,\frac{\sin\frac{n\pi}{\omega}y}{n}\right)\frac{4q^{2n}}{1-q^{2n}}\sin\frac{n\pi}{2\omega}(\psi+\beta_1). \qquad (68)$$

$$\frac{h_1}{\alpha^2 h_3}(c_0 t+\beta_3)=\left\{\frac{1}{\wp' z}\left(\zeta z-\frac{\eta z}{\omega}\right)-\frac{1}{\wp' y}\left(1-\frac{\wp'' y}{\wp'^2 y}\right)\left(\zeta y-\frac{\eta y}{\omega}\right)\right.$$

$$\left.+\frac{1}{\wp'^2 y}\left(\frac{\eta}{\omega}+\wp y\right)\right\}(\psi+\beta_1)$$

$$+\frac{\pi}{\wp'^2 y.\omega}\cdot\frac{\sin\frac{\pi}{2\omega}(\psi+\beta_1)}{\cos\frac{\pi y}{\omega}-\cos\frac{\pi}{2\omega}(\psi+\beta_1)}-\frac{1}{\wp' z}\log\frac{\sin\frac{\pi}{2\omega}\left(\frac{\psi+\beta_1}{2}+z\right)}{\sin\frac{\pi}{2\omega}\left(\frac{\psi+\beta_1}{2}-z\right)}$$

$$+\frac{1}{\wp' y}\left(1-\frac{\wp'' y}{\wp'^2 y}\right)\log\frac{\sin\frac{\pi}{2\omega}\left(\frac{\psi+\beta_1}{2}+y\right)}{\sin\frac{\pi}{2\omega}\left(\frac{\psi+\beta_1}{2}-y\right)}$$

$$+\sum_{n=1}^{\infty}\left\{-\frac{1}{\wp' z}\,\frac{\sin\frac{n\pi z}{\omega}}{n}+\frac{1}{\wp' y}\left(1-\frac{\wp'' y}{\wp'^2 y}\right)\frac{\sin\frac{n\pi y}{\omega}}{n}\right.$$

$$\left.+\frac{\pi}{\wp'^2 y.\omega}\cos\frac{n\pi}{\omega}y\right\}\frac{4q^{2n}}{1-q^{2n}}\sin\frac{n\pi}{2\omega}(\psi+\beta_1), \qquad (69)$$

$$u-\frac{1}{3}=-\frac{\eta}{\omega}+\left(\frac{\pi}{2\omega}\right)\frac{2}{1-\cos\frac{\pi}{2\omega}(\psi+\beta_1)}$$

$$-2\left(\frac{\pi}{\omega}\right)^2\cdot\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos\frac{n\pi}{2\omega}(\psi+\beta_1). \qquad (70)$$

From these values of s, u and t, we get

$$\left.\begin{aligned}&r=\frac{\alpha}{u},\\&\tan(\theta+\beta_2)=\tan\psi\cos I,\\&\cos\varphi=\sin I\sin\psi,\end{aligned}\right\}\tag{71}$$

and t. These constitute the required solution of our problem. The following Chapters are devoted to a thorough discussion of these solutions. Morton[1] has made a detailed study of the second equation of (49) after Forsyth, but he has limited his work to the form of the trajectories only and has left the variation of t out of account. Discussion of the second and third equations was done by Droste,[2] but very briefly. Hence neither of their works can be considered to have completely exhausted the necessary investigation.

Chapter V.

Distribution of the Singularities.

1. In oder to obtain the real expressions with real arguments of the series (68), (69) and (70), we at first discuss the distribution of the singularities among the roots of the fundamental cubic (39).

As we see from (63) and (65) remembering the properties of the σ-function of Weierstrass, the variables s and t admit[3]

$$(A):\qquad \frac{\psi+\beta_1}{2}\pm y\equiv 0,\quad (2\omega, 2\omega')$$

$$(B):\qquad \frac{\psi+\beta_1}{2}\pm z\equiv 0,\quad (2\omega, 2\omega')$$

as the infinity points and indeed as the logarithmic infinity points. From (54) r has an infinity point at $\wp\left(\frac{\psi+\beta_1}{2}\right)+\frac{1}{3}=0$, or $\wp\left(\frac{\psi+\beta_1}{2}\right)=\wp y$, or $\frac{\psi+\beta_1}{2}\pm y\equiv 0$ $(2\omega, 2\omega')$. This point coincides with the above singularity (A). Hence the only singularities are at (A) and at (B). The same

1) Morton, Phil. Mag. 42 (1921) 511. Also, Whittaker, *Analytical Dynamics*. Third Ed., did the same thing. Forsyth, Proc. Roy. Soc. London. 97 A (1920) 145; M. N. 82 (1921) 2, used Legendre's elliptic functions.

2) Droste, Proc. Acad. Amsterdam. 19 (1916) 197. Of course the deduction of these equations by these writers is quite different from ours and more elementary.

3) The following expressions are the congruence relations in the theory of numbers.

can be seen from (68), (69) and (70) that the point (A) behaves itself as a pole and a logarithmic infinity point, and the point (B) as a logarithmic infinity point. Mathematically the pole at (A) of (70) can be transformed away by a suitable substitution. The singularities which ought to be discussed are (A) and (B), provided that the three roots of the fundamental cubic e_1, e_2 and e_3 are all distinct.

Thus in (63) and (65),

if $\dfrac{h_3}{h_1\wp' y}\left(1-\dfrac{\wp'' y}{\wp'^2 y}\right)>0,$

then the infinity point for $t\to+\infty$ is

$$(A_2) \qquad \frac{\psi+\beta_1}{2}-y\equiv 0, \quad (2\omega, 2\omega'),$$

and the infinity point for $t\to-\infty$ is

$$(A_1) \qquad \frac{\psi+\beta_1}{2}+y\equiv 0, \quad (2\omega, 2\omega');$$

if $\dfrac{h_3}{h_1\wp' y}\left(1-\dfrac{\wp'' y}{\wp'^2 y}\right)<0,$

then the infinity point for $t\to+\infty$ is

$$(A_1) \qquad \frac{\psi+\beta_1}{2}+y\equiv 0, \quad (2\omega, 2\omega'),$$

and that for $t\to-\infty$ is

$$(A_2) \qquad \frac{\psi+\beta_1}{2}-y\equiv 0, \quad (2\omega, 2\omega').$$

If $\dfrac{h_3}{h_1\wp' z}<0,$

then the infinity point for $t\to+\infty$ is

$$(B_2) \qquad \frac{\psi+\beta_1}{2}-z\equiv 0, \quad (2\omega, 2\omega'),$$

and that for $t\to-\infty$ is

$$(B_1) \qquad \frac{\psi+\beta_1}{2}+z\equiv 0, \quad (2\omega, 2\omega');$$

if $\dfrac{h_3}{h_1\wp' z}>0,$

then the infinity point for $t\to+\infty$ is

$$(B_1) \qquad \frac{\psi+\beta_1}{2}+z\equiv 0, \quad (2\omega, 2\omega'),$$

and that for $t \to -\infty$ is

$$(B_2) \qquad \frac{\psi+\beta_1}{2}-z\equiv 0, \quad (2\omega, 2\omega').$$

The question is whether they are real or not. If they are real, the motion is asymptotic at these points, that is, the moving particle infinitely approaches to the point corresponding to (A) or to (B) as the time tends to $+\infty$ or $-\infty$. They are not logarithmic singularities for (70). Hence the trajectory of the moving particle in space is not necessarily asymptotic at these points. Such asymptotic character of the motion only occurs when the two of the roots of the fundamental cubic coincide, as will be shown in Chap. VII. Denote the singular value of ψ by $\bar{\psi}$.

For (A): $\bar{\psi}_{A1}\equiv-\beta_1-2y$, $(2\omega, 2\omega')$, or $\bar{\psi}_{A2}\equiv-\beta_1+2y$, $(2\omega, 2\omega')$;

for (B): $\bar{\psi}_{B1}\equiv-\beta_1-2z$, $(2\omega, 2\omega')$, or $\bar{\psi}_{B2}\equiv-\beta_1+2z$, $(2\omega, 2\omega')$.

The value of $\bar{\psi}$ to which the particle tends for $t\to+\infty$ is the same for the approach of ψ to $\bar{\psi}$ in the domain $\psi>\bar{\psi}$ as for the approach to the same point in the domain $\psi<\bar{\psi}$, that is, the same in the domain $r>\bar{r}$ as in the domain $r<\bar{r}$, where $\bar{r}$ denotes the corrosponding value of r for the singularity.

At (A): $$x=\wp y=-\frac{1}{3}, \quad u=0, \quad r=\infty;$$

at (B): $$x=\wp z=\frac{2}{3}, \quad u=1, \quad r=\alpha.$$

Thus the corresponding singularities for u are $u=0$ and $u=1$, and those for x are $x=-\dfrac{1}{3}$ and $x=\dfrac{2}{3}$.

2. Next we examine in which domain divided by the roots of the cubic (39) these singularities are located.

The fundamental cubic (39):

$$U(u)=u^3-u^2+\lambda u-\lambda(1-\mu) \tag{72}$$

with

$$u=\frac{\alpha}{r}, \quad \lambda=\frac{\alpha^2}{h_1^2} \quad \mu=h_3^2,$$

can be written, by (51), in the form:

$$X(x)=4x^3-g_2x-g_3=4(x-e_1)(x-e_2)(x-e_3), \tag{73}$$

with

$$g_2=4\left(\frac{1}{3}-\lambda\right), \quad g_3=4\left(\frac{2}{27}+\frac{2}{3}\lambda-\lambda\mu\right),$$

and

$$e_1+e_2+e_3=0, \quad e_1e_2+e_2e_3+e_3e_1=-\left(\frac{1}{3}-\lambda\right),$$

$$e_1e_2e_3=\frac{2}{27}+\frac{2}{3}\lambda-\lambda\mu.$$

At e_1: $\quad x=e_1, \quad u=u_1=e_1+\frac{1}{3}, \quad r=r_1=\frac{\alpha}{e_1+\frac{1}{3}}$;

at e_2: $\quad x=e_2, \quad u=u_2=e_2+\frac{1}{3}, \quad r=r_2=\frac{\alpha}{e_2+\frac{1}{3}}$;

at e_3: $\quad x=e_3, \quad u=u_3=e_3+\frac{1}{3}, \quad r=r_3=\frac{\alpha}{e_3+\frac{1}{3}}$.

At first we consider that any of these roots is distinct from others. Let us write the cubic (73) in the form:

$$f(x\,;\lambda,\mu)\equiv x^3-\left(\frac{1}{3}-\lambda\right)x-\left(\frac{2}{27}+\frac{2}{3}\lambda-\lambda\mu\right). \tag{74}$$

It ought to be remembered that λ and μ can not be negative.

(i) At $x=+\infty$: $\quad f(+\infty\,;\lambda,\mu)>0$;

at $x=\frac{2}{3}$: $\quad f\left(\frac{2}{3}\,;\lambda,\mu\right)=\lambda\mu>0$;

at $x=-\infty$: $\quad f(-\infty\,;\lambda,\mu)<0$.

Hence there are no or two real roots in the interval $\frac{2}{3}<x<+\infty$.

(ii) Consider a curve

$$(L): \quad y=f(x;0,\mu)\equiv x^3-\frac{1}{3}x-\frac{2}{27}\equiv\left(x+\frac{1}{3}\right)^2\left(x-\frac{2}{3}\right),$$

and a curve

$$(C): \quad y=f(x;\lambda,\mu)\equiv x^3-\left(\frac{1}{3}-\lambda\right)x-\left(\frac{2}{27}+\frac{2}{3}\lambda-\lambda\mu\right).$$

The curve (L) intersects the x-axis at $x=\frac{2}{3}$ and at $x=-\frac{1}{3}$ and touches it at $x=-\frac{1}{3}$. If we take the difference of the abscissae for these two curves, we get

$$y_C-y_L=\lambda\left(x-\frac{2}{3}+\mu\right),$$

where y_C and y_L represent the abscissae of the curves (C) and (L), respectively, for the same value of the ordinates. Hence the curve (C) lies above the curve (L) while $x>\frac{2}{3}-\mu$; and (C) lies beneath (L) while $x<\frac{2}{3}-\mu$; and (C) and (L) intersect at $x=\frac{2}{3}-\mu$. As $\mu>0$, the largest value of x, for which these two curves (C) and (L) intersect, is $+\frac{2}{3}$. Hence the intersection must occur for smaller values of x. Hence the curve (C) lies above (L) for $x>\frac{2}{3}$, whatever μ may be. Hence (C) can not intersect the x-axis for the values of x such that $x>\frac{2}{3}$. Thus in the interval $\frac{2}{3}<x<+\infty$ there is no real root of the cubic (74). (See, Fig. 1)

(iii) The ordinates of the extrema x_m of the curve (C) are obtained by

$$\frac{dy_C}{dx}=0, \quad i.\,e., \quad x_m=\pm\frac{\sqrt{1-3\lambda}}{3}.$$

As $\lambda>0$, $0<|x_m|<\frac{1}{3}$. Hence the double roots can be found only in the interval $-\frac{1}{3}<x<+\frac{1}{3}$. For $\lambda>\frac{1}{3}$, there occurs no extremum. Hence more than one root of the equation $f(x;\lambda,\mu)=0$ can not lie in the interval $-\infty<x<-\frac{1}{3}$. Hence one or no root is found in the interval $-\infty<x<-\frac{1}{3}$.

(iv) At $x=-\frac{1}{3}$, $f\left(-\frac{1}{3};\lambda,\mu\right)=-\lambda(1-\mu)$. If $\mu>1$, we can expect a real root of $f(x;\lambda,\mu)=0$ in the interval $-\infty<x<-\frac{1}{3}$. If $\mu<1$, there is no real root in this region.

(v) For $\lambda=\frac{1}{3}$ the curve (C) reduces to $y=x^3-\left(\frac{8}{27}-\frac{\mu}{3}\right)$. This has an inflection-point at $x=0$ and the tangent at that point is parallel to the x-axis. Of course there is only one real root in this case.

Especially for $\mu=\frac{8}{9}$, this inflection-point falls at the origin of the co-ordinates and the three roots of the cubic coincide at this point. This is the only point in which the three roots coincide, because the point of parallel inflection to the x-axis is one in which the three consecutive points have the same abscissae and the inflection of such nature can only take place on the y-axis.

From these considerations we infer that the possible cases of the distribution of the roots of the fundamental cubic (73) among the singularities are as follows:

$\Delta>0$, *i. e.*, there are three real roots: $e_3<e_2<e_1$.

Case I: $\mu>1$, then $e_3<-\frac{1}{3}<e_2<e_1<\frac{2}{3}$;

Case II: $\mu<1$, then $-\frac{1}{3}<e_3<e_2<e_1<\frac{2}{3}$;

$\Delta<0$. There is only one real root.

Case III: $\mu>1$, then $e_3<-\frac{1}{3}<\frac{2}{3}$;

Case IV: $\mu<1$, then $-\frac{1}{3}<e_k<\frac{2}{3}$. ($k=1$ or 3)

Fig. 1 shows the relative position of the various curves $y=f(x;\lambda,\mu)$ for different values of λ and μ, which correspond to these different cases of the distribution. Two important curves $y=f(x;\ o,\ \mu)$ and $y=f\left(x;\ \frac{1}{3},\ \mu\right)$ are also drawn in the figure.

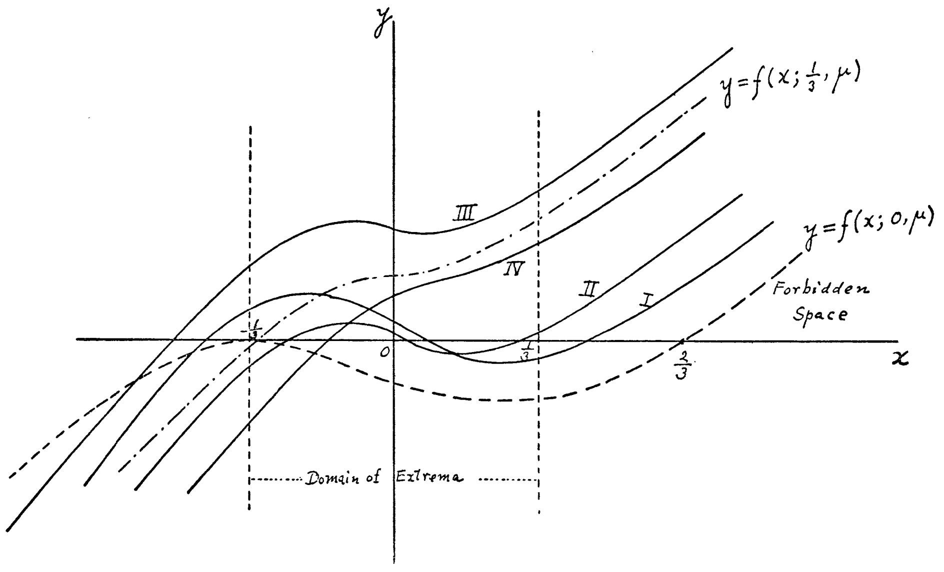

Fig. 1.

3. Our next step consists in the investigation of the sign of the

discriminant Δ of the fundamental cubic (73).

$$\Delta = -64\lambda\left[\lambda^2+\left(2-9\mu+\frac{27}{4}\mu^2\right)\lambda+(1-\mu)\right]. \tag{75}$$

If the quadratic in λ enclosed by the parenthesis in this expression has no real factor, then $\Delta<0$ for $\lambda>0$. For $\lambda=0$, $\Delta=0$. This is the case of the curve (L) in Fig. 1. Δ also vanishes for

$$\lambda^2+\left(2-9\mu+\frac{27}{4}\mu^2\right)\lambda+(1-\mu)=0. \tag{76}$$

Denote the two roots of this equation by λ_1 and λ_2.

$$\lambda_1,\ \lambda_2=-\left(1-\frac{9}{2}\mu+\frac{27}{8}\mu^2\right)\pm\sqrt{\left(1-\frac{9}{2}\mu+\frac{27}{8}\mu^2\right)^2-(1-\mu)}.$$

If
$$\left(1-\frac{9}{2}\mu+\frac{27}{3}\mu^2\right)^2-(1-\mu)\equiv\left(\frac{27}{8}\right)^2\mu\left(\mu-\frac{8}{9}\right)^3>0,$$

or if
$$\mu\left(\mu-\frac{8}{9}\right)^3>0,$$

or, as $\mu>0$, if
$$\frac{8}{9}<\mu,$$

then λ_1 and λ_2 are both real.

If $0<\mu<\frac{8}{9}$, then λ_1 and λ_2 are complex.

Suppose then that λ_1 and λ_2 are both real.

$$1-\frac{9}{2}\mu+\frac{27}{8}\mu^2<0 \text{ for } \frac{2}{3\sqrt{3}}(\sqrt{3}-1)<\mu<\frac{2}{3\sqrt{3}}(\sqrt{3}+1),$$
$$\text{or}\quad 0.275\ldots.<\mu<1.051\ldots.;$$

and
$$1-\frac{9}{2}\mu+\frac{27}{8}\mu^2>0 \quad\text{for}\quad \begin{cases} -\infty<\mu<\frac{2}{3\sqrt{3}}(\sqrt{3}-1), \\ \frac{2}{3\sqrt{3}}(\sqrt{3}+1)<\mu<+\infty. \end{cases}$$

Further according as $\mu \lesseqgtr 1$,

$$\left|1-\frac{9}{2}\mu+\frac{27}{8}\mu^2\right| \gtreqless \sqrt{\left(1-\frac{9}{2}\mu+\frac{27}{8}\mu^2\right)^2-(1-\mu)}.$$

Hence

for $0.888\ldots.<\mu<1$, λ_1 and λ_2 are both positive,

$$\left.\begin{array}{l}\text{for } 1<\mu<1.051\ldots,\\ \text{for } 1.051\ldots<\mu<\infty,\end{array}\right\}\ \text{one of the roots is positive.}$$

Thus we get the following schema:

Table I.

$0 \leqq \mu < 0.275 \ldots,$	λ_1, λ_2, complex,		$\Delta < 0,$	Case IV;
$0.275 \ldots < \mu < 0.888 \ldots,$	λ_1, λ_2 complex,		$\Delta < 0,$	Case IV;
		$0 < \lambda < \lambda_2,$	$\Delta < 0,$	Case IV;
$0.888 \ldots < \mu < 1,$	$0 < \lambda_2 < \lambda_1$, real,	$\lambda_2 < \lambda < \lambda_1,$	$\Delta > 0,$	Case II;
		$\lambda_1 < \lambda < \infty,$	$\Delta < 0,$	Case IV;
$1 < \mu < 1.051 \ldots,$	$\lambda_2 < 0 < \lambda_1$, real,	$0 < \lambda < \lambda_1,$	$\Delta > 0,$	Case I;
		$\lambda_1 < \lambda < \infty,$	$\Delta < 0,$	Case III;
$1.051 \ldots < \mu < +\infty,$	$\lambda_2 < 0 < \lambda_1$, real,	$0 < \lambda < \lambda_1,$	$\Delta > 0,$	Case I;
		$\lambda_1 < \lambda < \infty,$	$\Delta < 0,$	Case III.

In this schema the greater real root of the quadratic in λ of (76) is denoted by λ_1 and the lesser by λ_2.

$$\lambda_1, \lambda_2 = -\left(1 - \frac{9}{2}\mu + \frac{27}{8}\mu^2\right) \pm \frac{27}{8}\sqrt{\mu\left(\mu - \frac{8}{9}\right)^3}.$$

These values λ_1, λ_2 are determined by the value of μ. When μ is given, we can judge by this schema which case of the distribution can occur as we vary λ with that given value of μ.

4. We may proceed to the discussion of the sign of Δ in the following manner.

(75) can be written

$$\Delta = -64\lambda\left[\frac{27}{4}\lambda\mu^2 - (9\lambda + 1)\mu + (\lambda + 1)^2\right]. \tag{77}$$

$\Delta = 0$ for $\lambda = 0$ as before.

$\Delta = 0$ also for

$$\frac{27}{4}\lambda\mu^2 - (9\lambda + 1)\mu + (\lambda + 1)^2 = 0. \tag{78}$$

The roots μ_1, μ_2 of this quadratic are

$$\mu_1, \mu_2 = \frac{2}{27\lambda}\left\{(9\lambda + 1) \pm \sqrt{(9\lambda + 1)^2 - 27\lambda(\lambda + 1)^2}\right\}.$$

$$= \frac{2}{27\lambda}\left\{(9\lambda + 1) \pm \sqrt{-(3\lambda - 1)^3}\right\}.$$

If $\lambda < \frac{1}{3}$, μ_1 and μ_2 are both real and both positive,

because $|9\lambda + 1| > \sqrt{(9\lambda + 1)^2 - 27\lambda(\lambda + 1)^2}$;

If $\lambda > \frac{1}{3}$, μ_1 and μ_2 are complex.

Hence we get the following schema:

Table II.

$0<\lambda<\frac{1}{3}$,	$\mu_1>\mu_2>0$, real, $0<\mu<\mu_2$,	$\Delta<0$,	$\mu<1$,	Case IV;
			$\mu>1$,	Case III;
	$\mu_2<\mu<\mu_1$,	$\Delta>0$,	$\mu<1$,	Case II;
			$\mu>1$,	Case I;
	$\mu_1<\mu<\infty$,	$\Delta<0$,	$\mu<1$,	Case IV;
			$\mu>1$,	Case III;
$\frac{1}{3}<\lambda<\infty$,	μ_1, μ_2 complex,	$\Delta<0$,	$\mu<1$,	Case IV;
			$\mu>1$,	Case III.

By this schema we can judge which cases of the distribution can occur as we vary μ with a given value of λ.

5. Fig. 2 shows the curve $\Delta=0$ in the plane $\lambda\mu$. In the shaded region $\Delta>0$, while in the other region $\Delta<0$. The Roman characters in the Figure represent the cases of the distribution corresponding to these different domains of the values of λ and μ. The upper branch of the

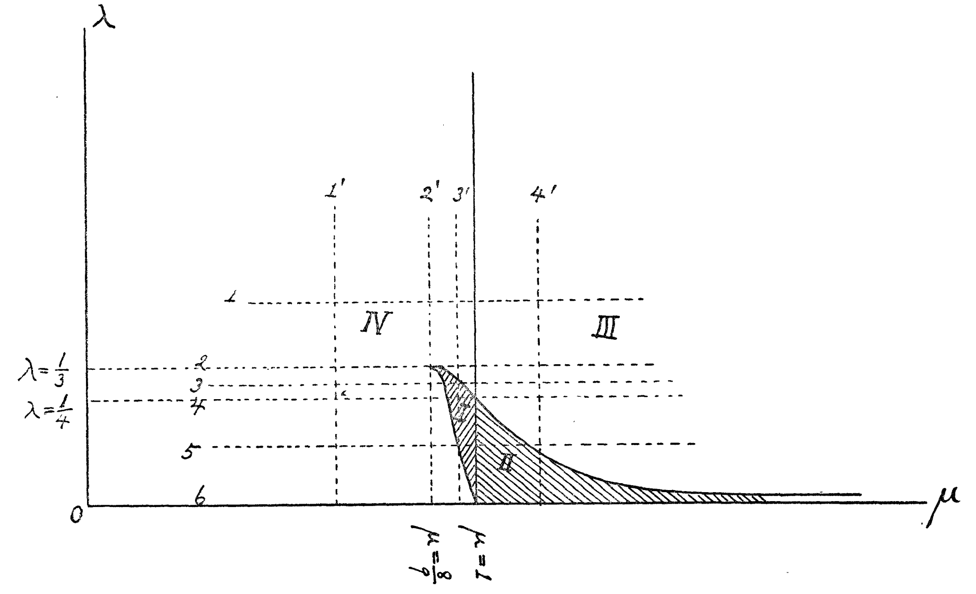

Fig. 2.

curve is asymptotic to the axis $\lambda=0$, as we can see from (75). There is a cusp at $\lambda=\frac{1}{3}$, $\mu=\frac{8}{9}$. At $\mu=1$ the lower branch of the curve intersects the λ-axis. $\mu=1$ is the boundary line for the Case I and the Case II, and also for the Case III and the Case IV.

The numerated lines parallel to the μ-axis show the mode of successive transitions between various cases of the distribution as we vary μ with a given λ.

1. $\frac{1}{3}<\lambda<\infty$,
2. $\frac{1}{3}=\lambda$,
3. $0<\lambda<\frac{1}{3}$, $\mu_2<\mu_1<1$,
4. $0<\lambda<\frac{1}{3}$, $\mu_2<\mu_1=1$,
5. $0<\lambda<\frac{1}{3}$, $\mu_2<1<\mu_1$,
6. $0\leqq\lambda<\frac{1}{3}$, $1=\mu_2<\mu_1$. This is the μ-axis.

The lines parallel to the λ-axis with dashed numerals show the mode of successive transitions between various cases of the distribution as we vary λ with a given μ.

1′. $0<\mu<0.888\ldots$,

2′. $0.888\ldots=\mu$. For $\lambda=\frac{1}{3}$ the three roots $e_1=e_2=e_3$.

3′. $0.888\ldots<\mu<1$,

4′. $1<\mu<\infty$.

The curve in the Figure represents the locus of the double roots as a function of λ and μ. The upper branch of the curve corresponds to $e_1=e_2$ and the lower to $e_2=e_3$. The cusp corresponds to the three equal roots $e_1=e_2=e_3$. Thus the three real roots exist in the shaded region, while e_1 or e_3 only exists in the unshaded region. The part of the region IV lying above the curve $\Delta=0$ between $\frac{8}{9}<\mu<1$ corresponds to the case in which there exists e_3 only and the part beneath to the case in which e_1 only exists. We return to this argument later when we discuss the correlation between the various possible types of motion.

6. To sum up, we get the following cases according to the various distributions of the roots of the cubic among the singularities (A) and(B) :

Table III.

Case I:	$e_3 < -\frac{1}{3} < e_2 < e_1 < \frac{2}{3}$,	$\mu > 1$,	(A) (B) both real;
Case II:	$-\frac{1}{3} < e_3 < e_2 < e_1 < \frac{2}{3}$,	$\mu < 1$,	(B) only real;
Case III:	$e_3 < -\frac{1}{3} < \frac{2}{3}$	$\mu > 1$,	(A) (B) both real;
Case IV:	$-\frac{1}{3} < e_k < \frac{2}{3}$, ($k=1$ or 3)	$\mu < 1$,	(B) only real.

The half double periods of the elliptic functions ω and ω' should be such that ω and $\frac{\omega'}{i}$ are both real, in order that we may get real values of the functions for real arguments. Moreover it does not restrict the generality of our discussion to assume that $\omega > 0$, $\frac{\omega'}{i} > 0$.

Thus the real domains of the elliptic functions are

(a): $e_1 < x < +\infty$,

and (b): $e_3 < x < e_2$.

In the cases (I) and (II) these two domains both exist, but in the cases (III) and (IV) there exits the domain (a) only. The singularity (A) is always in the real domain (a). The singularity (B) lies in the real domain (b) for (I) and in (a) for (III) and is imaginary for the cases (II) and (IV). According to which real domain we are considering, we distinguish sub-cases (a) and (b) in each of those cases. It hardly need to be mentioned that (b) exists only for the cases (I) and (II).

7. By the well-known properties[1] of the elliptic functions of such nature, we get the following table:

Table IV.

w	real argument	$\wp w$	$\wp' w$
$0 \ldots \omega$	w	$+\infty \ldots e_1$	real negative;
$\omega \ldots \omega + \omega'$	$w' = \frac{w - \omega}{i}$	$e_1 \ldots e_2$	positive imaginary;
$\omega + \omega' \ldots \omega'$	$w' = w - \omega'$	$e_2 \ldots e_3$	real positive;
$\omega' \ldots 0$	$w' = \frac{w}{i}$	$e_3 \ldots -\infty$	negative imaginary.

1) Appell et Lacour, *Principes de la Théorie des Fonctions Elliptiques et Applications.* (1922) Chap. III.

In the subsequent treatment we employ the real quantities with primes introduced in this table. The following formulae, together with (67), will be used in the transformations,[1] in order to write out the real expansions of (68) (69) and (70).

$$\left.\begin{aligned}
&\wp(w+\omega')=-\frac{\eta}{\omega}-2\left(\frac{\pi}{\omega}\right)^2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}\cos\frac{n\pi w}{\omega},\\
&\zeta(w+\omega')=\frac{\eta w}{\omega}+\eta'+2\frac{\pi}{\omega}\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\sin\frac{n\pi w}{\omega},\\
&\eta=\zeta\omega,\quad \eta'=\zeta\omega',\\
&\sigma(w\pm\omega')=\pm\sigma(\omega')\sigma_3(w)e^{\pm\eta' w},\\
&\log\sigma_3 w=\frac{\eta w^2}{2\omega}+2\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\frac{1-\cos\frac{n\pi w}{\omega}}{n},\\
&\log\frac{\sigma(w+\omega')}{\sigma(v+\omega')}=\log\frac{\sigma_3(w)}{\sigma_3(v)}+\eta'(w-v),\\
&\zeta(w+2\omega')=\zeta w+2\eta',\\
&\sigma(w+2\omega')=-\sigma(w)e^{2\eta'(w+\omega')},\\
&\zeta(iw)=-i\bar{\zeta}w,\\
&\bar{\zeta}(w)=\zeta(w\,;\,g_2,\,-g_3),\\
&\frac{\sigma(v+\omega')}{\sigma(w-\omega')}=-\frac{\sigma(v+\omega')}{\sigma(w+\omega')}e^{2\eta' w}.
\end{aligned}\right\}\quad(79)$$

Chapter VI.

Non-Degenerate Cases.

In this Chapter the real expressions in elementary functions with real arguments of our formulae (54) (63) and (65) are obtained in the cases when the roots of the fundamental cubic (73) are all distinct. In these cases we call *non-degenerate* according to the usual terminology in the theory of the elliptic functions. Otherwise we call *degenerate.* Degenerate cases are treated in the next Chapter.

(Ia) $$e_3<-\frac{1}{3}<e_2<e_1<\frac{2}{3},\qquad e_1<x<\infty,\ \mu>1.$$

1) Halphen, *loc. cit.* p. 426 and p. 140.

We put

$$\psi=\psi, \quad y=y'+\omega', \quad z=z, \quad h_3^2>1\,;$$
$$\beta_0=\beta_0'+i\beta_0'', \quad \beta_3=\beta_3'+i\beta_3'',$$

with real constants β_0', β_0'', β_3' and β_3''.

Then

$$\wp' y=+\frac{2\alpha}{h_1}\sqrt{h_3^2-1}, \quad \wp' z=-\frac{2\alpha h_3}{h_1}, \quad \wp'' y=\frac{2\alpha^2}{h_1^2}.$$

(54) (63) (65) change into:

$$\frac{2(h_3^2-1)}{h_1}(s+\beta'_0)$$
$$=\left\{\frac{1}{3}-\frac{\eta}{\omega}-\frac{2\alpha\pi}{\omega h_1\sqrt{h_3^2-1}}\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\sin\frac{n\pi y'}{\omega}\right\}(\psi+\beta_1)$$
$$-\sum_{n=1}^{\infty}\left\{\frac{4\pi}{\omega}\cos\frac{n\pi}{\omega}y' -\frac{4\alpha}{h_1\sqrt{h_3^2-1}}\frac{\sin\frac{n\pi}{\omega}y'}{n}\right\}\frac{q^n}{1-q^{2n}}\sin\frac{n\pi}{2\omega}(\psi+\beta_1),$$

$$\frac{2}{\alpha h_3}(c_0 t+\beta'_3)$$
$$=-\left\{\frac{1}{h_3}\left(\zeta z-\frac{\eta z}{\omega}\right)+\frac{2h_3^2-3}{2(h_3^2-1)^{\frac{3}{2}}}\cdot\frac{2\pi}{\omega}\left(\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\sin\frac{n\pi}{\omega}y'\right)\right.$$
$$\left.-\frac{h_1}{2\alpha(h_3^2-1)}\left(\frac{\eta}{\omega}-\frac{1}{3}\right)\right\}(\psi+\beta_1)$$
$$+\frac{1}{h_3}\log\frac{\sin\frac{\pi}{2\omega}\left(\frac{\psi+\beta_1}{2}+z\right)}{\sin\frac{\pi}{2\omega}\left(\frac{\psi+\beta_1}{2}-z\right)}$$
$$+\sum_{n=1}^{\infty}\left\{\frac{4q^n}{h_3}\frac{\sin\frac{n\pi}{\omega}z}{n}+\frac{2(2h_3^2-3)}{(h_3^2-1)^{\frac{3}{2}}}\frac{\sin\frac{n\pi}{\omega}y'}{n}\right.$$
$$\left.+\frac{2\pi h_1}{\alpha(h_3^2-1)\omega}\cos\frac{n\pi}{\omega}y'\right\}\frac{q^n}{1-q^{2n}}\sin\frac{n\pi}{2\omega}(\psi+\beta_1),$$

$$u-\frac{1}{3}=-\frac{\eta}{\omega}+\left(\frac{\pi}{2\omega}\right)^2\frac{2}{1-\cos\frac{\pi}{2\omega}(\psi+\beta_1)}$$

$$-2\left(\frac{\pi}{\omega}\right)^2\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos\frac{n\pi}{2\omega}(\psi+\beta_1),$$

$$\frac{2(h_3^2-1)}{h_1}.\ \beta_0{}''i=\frac{\alpha\pi}{h_1\sqrt{h_3^2-1}}.\ i,$$

$$\frac{2}{\alpha h_3}\beta_3{}''i=-\frac{2h_3^2-3}{2(h_3^2-1)^{\frac{3}{2}}}.\ \pi i.$$

The first series has no singularity.

The singularity in the third equation is not to be afraid of, for it can be reduced mathematically to an ordinary point by a suitable transformation. But physically speaking, there occurs a collision with the central point-mass.

The second series becomes logarithmically infinite at

$$\frac{\bar{\psi}_1+\beta_1}{2}-z\equiv 0\ (2\omega),\ \text{or}\ \bar{\psi}_1\equiv 2z-\beta_1\ (4\omega),\ \text{as}\ t\to+\infty\,;$$

$$\frac{\bar{\psi}_2+\beta_1}{2}+z\equiv 0\ (2\omega),\ \text{or}\ \bar{\psi}_2=-2z-\beta_1\ (4\omega),\ \text{as}\ t\to-\infty.$$

These are the only real singularities of our solution.

There are two types of motion to be distinguished:

$$(\alpha)\quad \frac{2}{3}<x\leqq\infty,\quad \text{or}\quad 0\leqq r<\alpha;$$

$$(\beta)\quad e_1\leqq x<\frac{2}{3},\quad \text{or}\quad \alpha<r\leqq\frac{\alpha}{e_1+\frac{1}{3}}.$$

In (α) the motion starts by ejection from the central point-mass and approaches asymptotically to $r=\alpha$, or starts asymptotically from $r=\alpha$ and reaches the central mass as a collisional orbit. (Fig. 3.)

For both these cases (α) and (β) the motion starts asymptotically from $r=\alpha$, $\bar{\psi}_2=-2z-\beta_1$ for $t\to-\infty$, and attains the maximum radius vector $r=\frac{\alpha}{e_1+\frac{1}{3}}$ at $\psi=-\beta_1$, and then approaches asymptotically backward toward $r=\alpha$, $\bar{\psi}_1=2z-\beta_1$ for $t\to+\infty$. Here the values of z, similarly the values of $\frac{\psi}{2}$ and y appearing in the later part of this Chapter, are determined only by congruence equations to 2ω. The additive integral multiples of 2ω are left undetermined. The possibility of an infinite number of the values of z, y and $\frac{\psi}{2}$ caused by this indeterminacy is avoided by making a convention that we should determine the integration constant β_1 so that the values of z, together with those of y and of $\frac{\psi}{2}$, are contained between o and 2ω, that is, by restricting ourselves to the fundamental parallelogram in the theory of the elliptic functions. This convention amounts to fixing beforehand the direction, from which the longitude is counted. Let this convention hold all through our argument.

The geometrical trajectory in space is periodic between $r=\frac{\alpha}{e_1+\frac{1}{3}}$ and $r=0$, provided that $r=0$ is a mathematical point.[1] The period, during which the radius vector passes through its initial value again in the same sense, is 4ω for ψ. During this interval u makes a complete revolution from infinity to infinity, reaching its maximum[2]:

$$e_1=\left(\frac{\pi}{2\omega}\right)^2\left\{\frac{2}{3}+16\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\right\}, \qquad (n=1, 3, 5, \ldots.)$$

for $\psi+\beta_1=2\omega$. Its appearance is thus periodic, but the point of the maximum radius vector constantly advances. The tangent at $r=0$ at the moment of returning to the point, is deflected by a constant amount. Let $4\omega=2p+\nu$, where p is a positive integer, which may be zero, and ν is a positive number less than 2π. Then the geometrical trajectory makes p complete revolutions and a fraction of one revolution round the central mass in one period between $r=0$ and the next return to $r=0$, and the advance of the point of the maximum radius vector, or the deflection of the tangents at $r=0$ at the consecutive returns to $r=0$, is ν in the angle. This quantity depends on the integration constants h_1 and h_3 through ω.

Mathematically, if 2ω and π are in the ratio of an irrational number, then the domain $0<r<\frac{\alpha}{e_1+\frac{1}{3}}$ is covered *everywhere densely*, except the point $r=0$, by the points of the geometrical trajectory. If 2ω and π are in the ratio of a rational number, the geometrical trajectory assumes the same position after a certain number of revolutions,

1) A similar argument can be seen in Newtonian mechanics in papers by Sundman as he discusses the analytical continuation of the motion of three bodies after a collision. Sundman, Acta Societatis Sc. Fennicae. **35** (1909) No. 9; Acta Math. **36** (1912) 105.

2) Halphen, *loc. cit.* p. 447.

which is the least common multiple of the denominator and the enumerator of that ratio. Then the geometrical trajectory is periodic. (Fig. 3 and Fig. 4.)

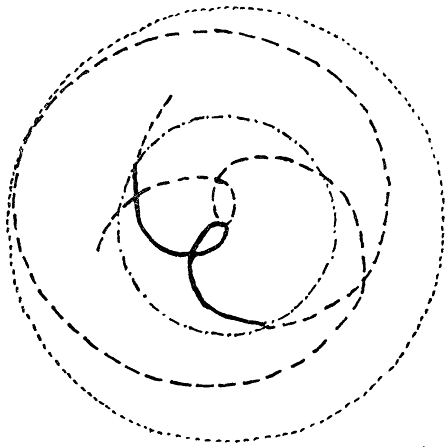

Fig. 3.
Inadmissible.

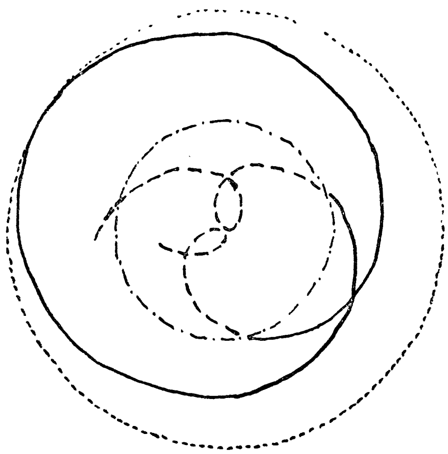

Fig. 4.
Pseudo-Elliptic.
$2\pi < 2\omega < 3\pi$
$2\pi < 4z < 3\pi$

However the actual or the physical trajectory is cut off at the points where the geometrical trajectory is intersected by the circle $r=\alpha$. The moving particle asymptotically approaches to one of these points: $\psi=\bar{\psi}_1$, $r=\alpha$ as $t\rightarrow+\infty$ and to the other: $\psi=\bar{\psi}_2$, $r=\alpha$ as $t\rightarrow-\infty$. This asymptotic approach is such that the velocity falls off continuously toward zero at this point, but not such that the particle performs an infinite number of revolutions near $r=\alpha$ as will be shown in some of the degenerate cases.

Thus the types of motion are divided into two: (α) and (β). But the type (α) is physically improbable.

In fact, it is quite improbable that in any star the distance $r=\alpha$ or $2m$ from the center lies outside its radius. In order that the radius of a star with its mass comparable with our Sun be equal to the distance $r=\alpha$, its density ought to be about 10^{17} times that of water,[1)]

1) For stars of different mass or of different radius this relation is of the form: (radius)^2 × (critical density) = constant, or (mass)^2 × (critical density) = constant; *i. e.*,

$$(\text{critical density}) = 10^{17} \times \frac{(\text{Sun's mass})^2}{(\text{mass of a star})^2},$$

or

$$(\text{critical density}) = 10^{17} \times \frac{(\text{Sun's radius})^2}{(\text{radius of a star})^2}.$$

while in the densest star, the companion of Sirius, a white dwarf, the density is about 6×10^4 times that of water.[1] There is no such diversity in the masses of the stars as to overcome this tremendous high magnitude of the critical density. Therefore the orbit inside $r=\alpha$ is physically highly improbable. Hence even the type (β) is physically a collisional orbit or an ejectional orbit,[2] because the radius $r=\alpha$ is situated completely inside the star.

To make the circumstance still worse, the type of motion (α) is physically inadmissible.

In fact, the coefficient of dt^2 of Schwarzschild's line element square in (20) vanishes at $r=\alpha$. This shows that the velocity of light becomes zero at this place. As r passes through this value, that coefficient changes its sign, the result being the same as to take t imaginary. This is inadmissible by the principle of relativity. Hence the motion for $r<\alpha$ should be excluded. The four dimensional manifold is, on the hyper-surface $r=\alpha$, flat in the axis of r and cylindrical in the axis of t. On the hyper-surface $r=0$ the circumstance is in the opposite. Any-

1) The most reasonable explanation will be that this is the limit of the relativistically possible density. This limit of physically possible density obtained from the Fermi-Dirac statistics in the theory of quantum mechanics may be less. Stoner, Phil. Mag. [vii]. **7** (1929) 63, gave the limit:

$$3.85 \times 10^6 \left(\frac{\text{mass of the Sun}}{\text{mass of a star}} \right)^2.$$

This is by far of low density than our limit. As to the last factor see the foot-note of the last page. Refer to: Pokrowski, Zeitscher. f. Phys. **49** (1928) 588, who gave the limit:

$$4 \times 10^{13}.$$

Anderson is of the opinion that, when the mass of a star is large enough, the density has no maximum. His consideration is based on the variability of the masses of the electrons as the density increases, and comes from the fact that the maximum density varies with the mass more rapidly than its square. He adds that this circumstance is in conformity with Milne's latest theory on the internal constitution of the stars. Nevertheless Anderson concludes that the density of the nucleus of a star can not have an infinitely great density from the cosmological point of view on the theory of relativity and that, in the case of our Sun, the limiting density ought to be 6.8×10^{17} times the density of water. This is of the same order of magnitude with our limit. Milne's view is to abondon the gas law in order to avoid the density from becoming infinitely great. Cf., W. Anderson, Z. f. Phys. **56** (1929) 854; **66** (1930) 280; E. A. Stoner, Phil. Mag. [vii] **9** (1930) 951; E. A. Milne, Nature **126** (1930) 238; **127** (1931) 16; Observatory. **53** (1930) 239; M. N. **91** (1930) 4.

2) A similar kind of argument can be seen in the problem of the collision of three bodies in the Newtonian mechanics. Cf., Levi-Civita, Acta Math. **30** (1906) 305; Armellini, Rendiconti. Atti Accademia Lincei. [v] **24** (1915) 184.

how the region $r<\alpha$ does not belong to our world of events.

The type of motion (β) may be said to be *pseudo-elliptic*. This type of motion is characterised by the property that its geometrical trajectory in space is such that the radius vector oscillates periodically between zero and a finite constant limit greater than α with the period different from that of the longitude and the point of the maximum radius vector constantly advances by the same amount, while the dynamical trajectory is the outer part of the geometrical trajectory cut off by a circle with radius $r=\alpha$ and is asymptotic for $t\to-\infty$ to one of these two intersections and also for $t\to+\infty$ to the other point of intersection. (Fig. 4.)

Suppose that e_1 and e_2 coincide. Then the domains of the two types of motion (Ia) and (Ib) have a common region $r=\dfrac{\alpha}{e_1+\frac{1}{3}}=\dfrac{\alpha}{e_2+\frac{1}{3}}$. This circle in space is the so-called *cycle limite*[1] in Poincaré's terminology, which means that possible types of motion approach to this circle from both sides asymptotically, each making an infinite number of revolutions round the central body. Hence the types of motion (Ia) (I b) together can be continued to the type (III a), when e_1 and e_2 become imaginary through this limiting type. e_3 ought to pass $x=-\dfrac{1}{3}$ in order that we get to the type (II b).

(II a) $$-\frac{1}{3}<e_3<e_2<e_1<\frac{1}{3},\quad e_1<x<\infty,\quad \mu<1.$$

We put

$$\psi=\psi,\quad y=iy',\quad z=z,\quad h_3^2<1\,;$$

then

$$\wp' z=-i\frac{2\alpha}{h_1}\sqrt{1-h_3^2},\quad \wp' z=-\frac{2\alpha h_3}{h_1},\quad \wp'' y=\frac{2\alpha^2}{h_1^2}.$$

$$\frac{2(h_3^2-1)}{h_1}(s+\beta_0)=\left\{\frac{1}{3}-\frac{\eta}{\omega}-\frac{\alpha}{h_1\sqrt{1-h_3^2}}\left(\frac{\eta y'}{\omega}+\bar{\zeta}y'\right)\right\}(\psi+\beta_1)$$

$$-\frac{\pi}{\omega}\,\frac{\sin\dfrac{\pi}{2\omega}(\psi+\beta_1)}{\cosh\dfrac{\pi}{\omega}y'-\cos\dfrac{\pi}{2\omega}(\psi+\beta_1)}$$

1) Poincaré, Journal de Math. [iii] 8 (1882) 251. A detailed consideration is found later.

$$-\frac{\alpha}{h_1\sqrt{1-h_3^2}}\cdot \tan^{-1}\left\{\frac{\sinh\frac{\pi}{\omega}\, y' \sin\frac{\pi}{2\omega}(\psi+\beta_1)}{1-\cosh\frac{\pi}{\omega}\, y' \cos\frac{\pi}{2\omega}(\psi+\beta_1)}\right\}$$

$$+\sum_{n=1}^{\infty}\left\{\frac{4\pi}{\omega}\cosh\frac{n\pi}{\omega}y'\right.$$

$$\left.-\frac{4\alpha}{h_1\sqrt{1-h_3^2}}\,\frac{\sinh\frac{n\pi}{\omega}y'}{n}\right\}\frac{q^{2n}}{1-q^{2n}}\sin\frac{n\pi}{2\omega}(\psi+\beta_1),$$

$$\frac{2}{\alpha h_3}(c_0t+\beta_3)=\left\{-\frac{1}{h_3}\left(\zeta z-\frac{\eta z}{\omega}\right)+\frac{2h_3^2-3}{2(1-h_3^2)^{\frac{3}{2}}}\left(\zeta y'+\frac{\eta y'}{\omega}\right)\right.$$

$$\left.+\frac{h_1}{2\alpha(h_3^2-1)}\left(\frac{\eta}{\omega}-\frac{1}{3}\right)\right\}(\psi+\beta_1)$$

$$+\frac{h_1\pi}{2\alpha(h_3^2-1)\,\omega}\cdot\frac{\sin\frac{\pi}{2\omega}(\psi+\beta_1)}{\cosh\frac{\pi}{\omega}y'-\cos\frac{\pi}{2\omega}(\psi+\beta_1)}$$

$$+\frac{1}{h_3}\log\frac{\sin\frac{\pi}{2\omega}\left(\frac{\psi+\beta_1}{2}+z\right)}{\sin\frac{\pi}{2\omega}\left(\frac{\psi+\beta_1}{2}-z\right)}$$

$$+\frac{2h_3^2-3}{2(1-h_3^2)^{\frac{3}{2}}}\tan^{-1}\left\{\frac{\sinh\frac{\pi}{\omega}\, y' \sin\frac{\pi}{2\omega}(\psi+\beta_1)}{1-\cosh\frac{\pi}{\omega}\, y' \cos\frac{\pi}{2\omega}(\psi+\beta_1)}\right\}$$

$$+\sum_{n=1}^{\infty}\left\{\frac{4}{h_3}\,\frac{\sin\frac{n\pi}{\omega}z}{n}+\frac{2(2h_3^2-3)}{(1-h_3^2)^{\frac{3}{2}}}\cdot\frac{\sinh\frac{n\pi}{\omega}y'}{n}\right.$$

$$\left.+\frac{2\pi h_1}{\alpha(h_3^2-1)\omega}\cosh\frac{n\pi}{\omega}y'\right\}\frac{q^{2n}}{1-q^{2n}}\sin\frac{n\pi}{2\omega}(\psi+\beta_1),$$

$$u-\frac{1}{3}=-\frac{\eta}{\omega}+\left(\frac{\pi}{2\omega}\right)^2\cdot\frac{2}{1-\cos\frac{\pi}{2\omega}(\psi+\beta_1)}$$

$$-2\left(\frac{\pi}{\omega}\right)^2 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} \cos\frac{n\pi}{2\omega}(\psi+\beta_1).$$

This case is very similar to (Ia) and scarcely needs any description. It is *pseudo-elliptic*. The geometrical trajectory oscillates between $r=0$ and $r=\dfrac{\alpha}{e_1+\frac{1}{3}}$, just like in the case (Ia). (Fig. 3 and Fig. 4.). The advance of the point of the maximum radius vector and the deflection of the tangent at $r=0$ at the consecutive returns to $r=0$, are both in the sense of the motion of the moving particle, that is, direct, because $2\omega>\pi$. If $(2p+1)\pi<2\omega<(2p+2)\pi$, but near to 2π, then it looks as though the point of the maximum radius vector were in a retrograde motion. To see the relation $2\omega>\pi$, we start from the definition of the modulus of the elliptic integral :[1]

$$K=\int_0^1 \frac{dx}{\sqrt{(1-k^2x^2)(1-x^2)}}.$$

As $k^2<1$, we have[2]

$$K>\int_0^1 \frac{dx}{\sqrt{1-x^2}}=\frac{\pi}{2}.$$

$$\therefore\ K>\frac{\pi}{2}.$$

But[2]

$$\frac{K}{\sqrt{e_1-e_3}}=\omega.$$

So long as $e_1-e_3<1$, $\dfrac{2\omega}{\pi}<1$.

Now the relation $e_1-e_3<1$ holds for the cases (II) and (IV). Hence, in the cases (II) and (IV), the advance of the point of the maximum radius vector and the point of the minimum radius vector, if either or both of them exist, are always in the sense of the motion of the moving particle. This holds not only in the cases (IIa) (IVa) but also in the cases (II b) (IV b).

Two types of motion in this case can be distinguished :

1) This inequality can be seen also from

$$\sqrt{\frac{2K}{\pi}}=\Theta_1(o)=1+2q+2q^4+2q^9+\dots$$

Cf., Appell et Lacour, *loc. cit.* p. 140.

2) Appell et Lacour, *loc. cit.* p. 164.

(α) $\frac{2}{3}<x\leqq\infty$, inadmissible. Fig. 3.

(β) $e_1\leqq x<\frac{2}{3}$. The motion is asymptotic at $r=\alpha$.

The corresponding value of ψ is the same as in the type (Ia). Fig. 4.

If e_1 and e_2 coincide, the circle $r=\dfrac{\alpha}{e_1+\frac{1}{3}}=\dfrac{\alpha}{e_2+\frac{1}{3}}$ is a *cycle limite* and the type of motion changes into (IIa).

(III a) $$e_3<-\frac{1}{3}<\frac{2}{3}. \qquad e_3<x<\infty. \quad \mu>1.$$

Put

$$\psi=\psi, \quad y=y, \quad z=z, \quad h_3^2>1,$$

then

$$\wp' y=-\frac{2\alpha}{h_1}\sqrt{h_3^2-1}\,, \quad \wp' z=-\frac{2\alpha h_3}{h_1}, \quad \wp'' y=\frac{2\alpha^2}{h_1^2}.$$

$$\frac{2(h_3^2-1)}{h_1}(s+\beta_0)=\left\{\frac{1}{3}-\frac{\eta}{\omega}-\frac{\alpha}{h_1\sqrt{h_3^2-1}}\left(\frac{y\eta}{\omega}-\zeta y\right)\right\}(\psi+\beta_1)$$

$$-\frac{\pi}{\omega}\;\frac{\sin\frac{\pi}{2\omega}(\psi+\beta_1)}{\cos\frac{\pi y}{\omega}-\cos\frac{\pi}{2\omega}(\psi+\beta_1)}$$

$$+\frac{\alpha}{h_1\sqrt{h_3^2-1}}\log\frac{\sin\frac{\pi}{2\omega}\left(\frac{\psi+\beta_1}{2}+y\right)}{\sin\frac{\pi}{2\omega}\left(\frac{\psi+\beta_1}{2}-y\right)}$$

$$+\sum_{n=1}^{\infty}\left\{\frac{4\pi}{\omega}\cos\frac{n\pi}{\omega}y-\frac{4\alpha}{nh_1\sqrt{h_3^2-1}}\sin\frac{n\pi}{\omega}y\right\}\frac{q^{2n}}{1-q^{2n}}\sin\frac{n\pi}{2\omega}(\psi+\beta_1),$$

$$\frac{2}{\alpha h_3}(c_0t+\beta_3)=\left\{-\frac{1}{h_3}\left(\zeta z-\frac{\eta z}{\omega}\right)+\frac{2h_3^2-3}{2(h_3^2-1)^{\frac{3}{2}}}\left(\zeta y-\frac{\eta y}{\omega}\right)\right.$$
$$\left.+\frac{h_1}{2\alpha(h_3^2-1)}\left(\frac{\eta}{\omega}-\frac{1}{3}\right)\right\}(\psi+\beta_1)$$

$$+\frac{h_1\pi}{2\alpha(h_3^2-1)\omega}\cdot\frac{\sin\frac{\pi}{2\omega}(\psi+\beta_1)}{\cos\frac{\pi y}{\omega}-\cos\frac{\pi}{2\omega}(\psi+\beta_1)}$$

$$+\frac{1}{h_3}\log\frac{\sin\frac{\pi}{2\omega}\left(\frac{\psi+\beta_1}{2}+z\right)}{\sin\frac{\pi}{2\omega}\left(\frac{\psi+\beta_1}{2}-z\right)}$$

$$-\frac{2h_3^2-3}{2(h_3^2-1)^{\frac{3}{2}}}\log\frac{\sin\frac{\pi}{2\omega}\left(\frac{\psi+\beta_1}{2}+y\right)}{\sin\frac{\pi}{2\omega}\left(\frac{\psi+\beta_1}{2}-y\right)}$$

$$+\sum_{n=1}^{\infty}\left\{\frac{4}{h_3}\frac{\sin\frac{n\pi}{\omega}z}{n}-\frac{2(2h_3^2-3)}{(h_3^2-1)^{\frac{3}{2}}n}\sin\frac{n\pi}{\omega}y\right.$$

$$\left.+\frac{2\pi h_1}{\alpha(h_3^2-1)\omega}\cos\frac{n\pi}{\omega}y\right\}\frac{q^{2n}}{1-q^{2n}}\sin\frac{n\pi}{2\omega}(\psi+\beta_1),$$

$$u-\frac{1}{3}=-\frac{\eta}{\omega}+\left(\frac{\pi}{2\omega}\right)^2\frac{2}{1-\cos\frac{\pi}{2\omega}(\psi+\beta_1)}$$

$$-2\left(\frac{\pi}{\omega}\right)^2\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos\frac{n\pi}{2\omega}(\psi+\beta_1).$$

The motion is asymptotic at (A) and (B). The geometrical trajectory in space at (B) is not asymptotic. There are three types of motion which can be imagined.

(α) $\frac{2}{3}<x\leqq\infty$. The motion is to occur from $r=0$ to $r=\alpha$ and asymptotic at $r=\alpha$. This type is similar to the types (I a α) (II a α) and physically inadmissible. (Fig. 5.).

(γ) $-\frac{1}{3}<x\leqq\frac{\alpha}{e_3+\frac{1}{3}}$. This is impossible because $r<0$.

(β) $-\frac{1}{3}<x<\frac{2}{3}$. The motion occurs from $r=\alpha$ to $r=\infty$ and asymptotic at both of these points. (Fig. 6.).

Consider the only possible type (β).

At $r=\alpha$:

$$\bar{\psi}_{B1}=-\beta_1-2z, \qquad \text{as } t\to-\infty;$$
$$\bar{\psi}_{B2}=-\beta_1+2z, \qquad \text{as } t\to+\infty.$$

At $r=\infty$:

$$\overline{\psi}_{A1}=-\beta_1-2y,\begin{cases}\text{for } t\to-\infty, & \text{if } 1<h_3^2<\dfrac{3}{2}\,;\\ \text{for } t\to+\infty, & \text{if } \dfrac{3}{2}<h_3^2\,;\end{cases}$$

$$\overline{\psi}_{A2}=-\beta_1+2y,\begin{cases}\text{for } t\to-\infty, & \text{if } \dfrac{3}{2}<h_3^2\,;\\ \text{for } t\to+\infty, & \text{if } 1<h_3^2<\dfrac{3}{2}.\end{cases}$$

These give the asymptotes for the trajectory in space for $r\to\infty$. The geometrical trajectory starts asymptotically from either of these asymptotes according to the sign of $h_3^2-\dfrac{3}{2}$ and reaches the origin at $\psi=-\beta_1$, and then tends to infinity asymptotically along the other asymptote. The trajectory is symmetrical with respect to $\psi=-\beta_1$. This geometrical trajectory turns by an angle $4y$ during two consecutive asymptotic approaches to infinity. If $4y=2p\pi+\nu$, where p is an integer, positive or zero, and ν is a positive number less than 2π, then the trajectory performs p revolutions and a fraction of one revolution during two consecutive asymptotic approaches to infinity. (Fig. 5 and Fig. 6.)

The actual trajectory is this geometrical trajectory cut by a circle $r=\alpha$ and these two points of intersection are the points of asymptotic approach as the time tends to $+\infty$ or $-\infty$. Hence the particle moves either from $\psi=\overline{\psi}_{B1}$, $r=\alpha$ to $\psi=\overline{\psi}_{A1}$, $r=\infty$ if $h_3^2-\dfrac{3}{2}>0$, and to $\psi=\overline{\psi}_{A2}$,

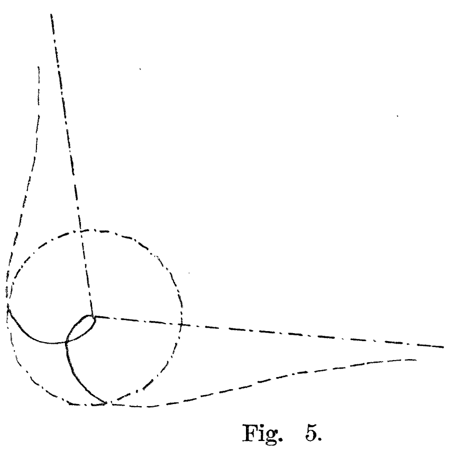

Fig. 5.
Inadmissible.

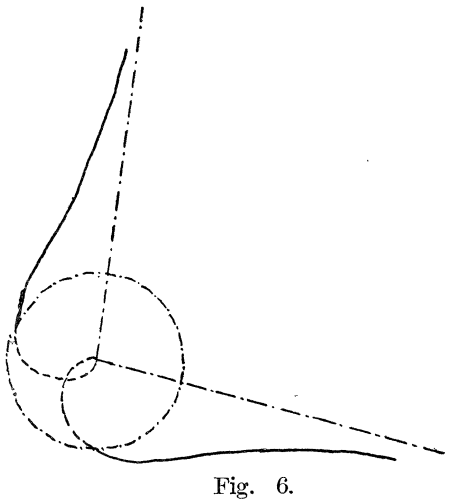

Fig. 6.
Pseudo-Hyperbolic.
$2\pi<4y<3\pi$ $2\pi<4z<3\pi$

$r=\infty$ if $h_3^2-\frac{3}{2}<0$; or from $\psi=\overline{\psi}_{A2}$, $r=\infty$ if $h_3^2-\frac{3}{2}>0$, and from $\psi=\psi_{A1}$, $r=\infty$ if $h_3^2-\frac{3}{2}<0$, to $\psi=\overline{\psi}_{B2}$, $r=\alpha$. All these approaches are asymptotic. This type of motion may be called *pseudo-hyperbolic*. We call *hyperbolic*, because the moving particle approaches asymptotically to infinity along two distinct asymptotes, one for $t\to-\infty$ and the other for $t\to+\infty$. *Pseudo* has the same meaning as in pseudo-elliptic. (Fig. 6.)

Especially, if e_3 coincides with $x=-\frac{1}{3}$, that is, $r=\frac{\alpha}{e_3+\frac{1}{3}}$ becomes infinite, then we call the type of motion *pseudo-parabolic*. In this case $y=0$ and the two asymptotes, $\psi=\overline{\psi}_{A1}$ and $\psi=\overline{\psi}_{A2}$, coincide. Hence it can properly be called *parabolic* in analogy with that in the Newtonian mechanics.

The type of motion can be continued to (IVa) when e_3 passes over to $x=-\frac{1}{3}$. If e_1 and e_2 come into real existence, then the type passes into the type (Ia) through the state with a *cycle limite*.

(IVa) $-\frac{1}{3}<e_k<\frac{2}{3}$, $(k=1 \text{ or } 3)$, $e_k<x<\infty$, $\mu<1$.

Put

$$\psi=\psi,\quad y=iy',\quad z=z,\quad h_3^2<1,$$

then

$$\wp' y=-i\frac{2\alpha}{h_1}\sqrt{1-h_3^2},\quad \wp' z=-\frac{2\alpha h_3}{h_1},\quad \wp'' y=\frac{2\alpha^2}{h_1^2}.$$

$$\frac{2(h_3^2-1)}{h_1}(s+\beta_0)=\left\{\frac{1}{3}-\frac{\eta}{\omega}-\frac{\alpha}{h_1\sqrt{h_3^2-1}}\left(\frac{y'\eta}{\omega}+\zeta y'\right)\right\}(\psi+\beta_1)$$

$$-\frac{\pi}{\omega}\frac{\sin\frac{\pi}{2\omega}(\psi+\beta_1)}{\cosh\frac{\pi}{\omega}y'-\cos\frac{\pi}{2\omega}(\psi+\beta_1)}$$

$$-\frac{\alpha}{h_1\sqrt{h_3^2-1}}\tan^{-1}\left\{\frac{\sinh\frac{\pi}{\omega}y'\sin\frac{\pi}{2\omega}(\psi+\beta_1)}{1-\cosh\frac{\pi}{\omega}y'\cos\frac{\pi}{2\omega}(\psi+\beta_1)}\right\}$$

$$+\sum_{n=1}^{\infty}\left\{\frac{4\pi}{\omega}\cosh\frac{n\pi}{\omega}y'-\frac{4\alpha}{nh_1\sqrt{1-h_3^2}}\sinh\frac{n\pi}{\omega}y'\right\}\frac{q^{2n}}{1-q^{2n}}\sin\frac{n\pi}{2\omega}(\psi+\beta_1),$$

$$\frac{2}{\alpha h_3}(c_0t+\beta_3)=\left\{-\frac{1}{h_3}\left(\zeta z-\frac{\eta z}{\omega}\right)+\frac{2h_3^2-3}{2(1-h_3^2)^{\frac{3}{2}}}\left(\zeta y'+\frac{\eta y'}{\omega}\right)\right.$$
$$\left.+\frac{h_1}{2\alpha(h_3^2-1)}\left(\frac{\eta}{\omega}-\frac{1}{3}\right)\right\}(\psi+\beta_1)$$
$$+\frac{h_1\pi}{2\alpha(h_3^2-1)\omega}\frac{\sin\frac{\pi}{2\omega}(\psi+\beta_1)}{\cosh\frac{\pi}{\omega}y'-\cos\frac{\pi}{2\omega}(\psi+\beta_1)}+\frac{1}{h_3}\log\frac{\sin\frac{\pi}{2\omega}\left(\frac{\psi+\beta_1}{2}+z\right)}{\sin\frac{\pi}{2\omega}\left(\frac{\psi+\beta_1}{2}-z\right)}$$
$$+\frac{2h_3^2-3}{2(1-h_3^2)^{\frac{3}{2}}}\tan^{-1}\left\{\frac{\sinh\frac{\pi}{\omega}y'\sin\frac{\pi}{2\omega}(\psi+\beta_1)}{1-\cosh\frac{\pi}{\omega}y'\cos\frac{\pi}{2\omega}(\psi+\beta_1)}\right\}$$
$$+\sum_{n=1}^{\infty}\left\{\frac{4}{h_3}\frac{\sin\frac{n\pi}{\omega}z}{n}+\frac{2(2h_3^2-3)}{(1-h_3^2)^{3/2}}\frac{\sinh\frac{n\pi}{\omega}y'}{n}\right.$$
$$\left.+\frac{2\pi h_1}{\alpha(h_3^2-1)\omega}\cosh\frac{n\pi}{\omega}y'\right\}\frac{q^{2n}}{1-q^{2n}}\sin\frac{n\pi}{2\omega}(\psi+\beta_1),$$
$$u-\frac{1}{3}=-\frac{\eta}{\omega}+\left(\frac{\pi}{2\omega}\right)^2\cdot\frac{2}{1-\cos\frac{\pi}{2\omega}(\psi+\beta_1)}$$
$$-2\left(\frac{\pi}{\omega}\right)^2\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos\frac{n\pi}{2\omega}(\psi+\beta_1).$$

There are two types of motion which can be imagined.

(α) $0\leqq r<\alpha$, Inadmissible. Fig. 3.

(β) $\alpha<r\leqq\dfrac{\alpha}{e_k+\frac{1}{3}}$. The motion is asymptotic at $r=\alpha$ as in (IIIa), but the singularity (A) does not exist. The motion is similar to (Ia) and (IIa) and is called *pseudo-elliptic*. Fig. 4.

The type of motion can be continued either to (IIIa) by making e_1 pass through $x=-\frac{1}{3}$, or to (IIa) by making e_1 and e_2 appear as real. If e_2 and e_3 coincide and then disappear in the type (IIa), then this type (IVa) will appear. At that moment the only real root is considered to be e_1. But through various transitions it may change into e_3 and crosses $x=-\frac{1}{3}$ to the other side.

(IIb)

$$-\frac{1}{3}<e_3<e_2<e_1<\frac{2}{3},\quad e_3<x<e_2,\quad \mu<1.$$

Put
$$\psi=\psi'+2\omega',\quad y=iy',\quad z=z,\quad h_3^2<1,$$
$$\beta_0=\beta_0'+i\beta_0'',\quad \beta_3=\beta_3'+i\beta_3'',$$

where $\beta_0', \beta_0'', \beta_3'$ and β_3'' are real constants. Then

$$\wp' y=-i\frac{2\alpha}{h_1}\sqrt{1-h_3^2},\quad \wp' z=-\frac{2\alpha h_3}{h_1},\quad \wp'' y=\frac{2\alpha^2}{h_1^2}.$$

$$\frac{2(h_3^2-1)}{h_1}(s+\beta'_0)=\left\{\frac{1}{3}-\frac{\eta}{\omega}-\frac{\alpha}{h_1\sqrt{1-h_3^2}}\left(\frac{\eta y'}{\omega}+\bar{\zeta}y'\right)\right\}(\psi'+\beta_1)$$
$$-\sum_{n=1}^{\infty}\left\{\frac{4\pi}{\omega}\cosh\frac{n\pi y'}{\omega}+\frac{4\alpha}{h_1\sqrt{1-h_3^2}}\frac{\sinh\frac{n\pi}{\omega}y'}{n}\right\}\frac{q^n}{1-q^{2n}}\sin\frac{n\pi}{2\omega}(\psi'+\beta_1),$$

$$\frac{2}{\alpha h_3}(c_0t+\beta'_3)=\left\{-\frac{1}{h_3}\left(\zeta z-\frac{\eta z}{\omega}\right)+\frac{2h_3^2-3}{2(1-h_3^2)^{3/2}}\left(\bar{\zeta}y'+\frac{\eta}{\omega}y'\right)\right.$$
$$\left.+\frac{h_1}{2\alpha(h_3^2-1)}\left(\frac{\eta}{\omega}-\frac{1}{3}\right)\right\}(\psi'+\beta_1)$$
$$+\sum_{n=1}^{\infty}\left\{\frac{4}{h_1}\frac{\sin\frac{n\pi}{\omega}z}{n}-\frac{2(2h_3^2-3)}{(1-h_3^2)^{3/2}}\frac{\sinh\frac{n\pi}{\omega}y'}{n}\right.$$
$$\left.+\frac{2\pi h_1}{\alpha(h_3^2-1)\omega}\cosh\frac{n\pi}{\omega}y'\right\}\frac{q^n}{1-q^{2n}}\sin\frac{n\pi}{2\omega}(\psi'+\beta_1),$$

$$u-\frac{1}{3}=-\frac{\eta}{\omega}-2\left(\frac{\pi}{\omega}\right)^2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}\cos\frac{n\pi}{2\omega}(\psi'+\beta_1),$$

$$-i\beta_0''=\frac{h_1}{h_3^2-1}\left\{\eta'+\frac{\omega'}{3}-\frac{\alpha\omega'}{h_1\sqrt{1-h_3^2}}\bar{\zeta}y'\right\},$$

$$\frac{2i}{\alpha h_3}\beta_3''=-2\frac{\omega'\zeta z-\eta' z}{h_3}+\frac{2h_3^2-3}{(1-h_3^2)^{3/2}}(\omega'\bar{\zeta}y'+\eta' y')+\frac{h_1}{\alpha(h_3^2-1)}\left(\frac{\omega'}{3}+\eta'\right).$$

The motion has no singularity in the whole duration of motion. It is periodic in the interval $e_3\leqq x\leqq e_2$ or $\frac{\alpha}{e_2+\frac{1}{3}}\leqq r\leqq\frac{\alpha}{e_3+\frac{1}{3}}$. The period of the motion with respect to r is 4ω for ψ', and

$$\frac{2\omega h_1}{h_3^2-1}\left\{\frac{1}{3}-\frac{\eta}{\omega}-\frac{\alpha}{h_1(1-h_3^2)^{3/2}}\left(\frac{\eta y'}{\omega}+\bar{\zeta}y'\right)\right\}\qquad\text{for } s;$$

and
$$\frac{2\omega\alpha h_3}{c_0}\left\{-\frac{1}{h_3}\left(\zeta z-\frac{\eta z}{\omega}\right)+\frac{2h_3^2-3}{2(1-h_3^2)^{3/2}}\left(\bar{\zeta}y'+\frac{y'\eta}{\omega}\right)+\frac{h_1}{2\alpha(h_3^2-1)}\left(\frac{\eta}{\omega}-\frac{1}{3}\right)\right\}\quad\text{for } t\,;$$

where
$$\zeta z-\frac{\eta z}{\omega}=\frac{\pi}{2\omega}\cot\frac{\pi z}{2\omega}+\frac{2\pi}{\omega}\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin\frac{n\pi z}{\omega},$$
$$\bar{\zeta}y'+\frac{\eta y'}{\omega}=\frac{\pi}{2\omega}\coth\frac{\pi y'}{2\omega}-\frac{2\pi}{\omega}\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sinh\frac{n\pi y'}{\omega},$$
$$\frac{\eta}{\omega}=\frac{1}{12}\left(\frac{\pi}{\omega}\right)^2-2\left(\frac{\pi}{\omega}\right)^2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2},\quad (n=1,3,5,\dots)$$
$$q=e^{-\frac{\pi\omega'}{\omega i}}.$$

Whether the motion is retrograde or direct can not be decided at this stage. It depends on the values of z and of y. If $\cot\frac{\pi z}{2\omega}$ is negative and great and $\coth\frac{\pi y'}{2\omega}$ is also negative and great, then the motion is evidently direct. It will be shown later in Chap. X that the sense of the motion is always direct. The point of the maximum radius vector always advances by a constant amount. Fig. 7.

Let $4\omega=2p\pi+\nu$, where p is an integer, positive or zero, and ν is a positive number less than 2π, then the particle performs p revolutions and a fraction of one revolution before passing the same value of r again in the same sense, especially for travelling from a perihelion to the immediately following perihelion.[1] If $\pi<\nu<2\pi$, then it looks as though the perihelion maintained a retrograde motion, but in reality it always advances. As h_1 and h_3 so vary that $\sqrt[4]{e_1-e_2}$ or $\sqrt[4]{e_1-e_3}$ approaches to unity, then the integer p tends to zero, because[2]
$$\sqrt{\frac{2\omega}{\pi}}\sqrt[4]{e_1-e_2}=1-2q+2q^4-2q^9+\dots,$$
$$\sqrt{\frac{2\omega}{\pi}}\sqrt[4]{e_1-e_3}=1+2q+2q^4+2q^9+\dots,$$
with $q<1$. In this way, if $p=0$ and ν is small enough, then the motion is the one deduced by Einstein, Schwarzschild and de Sitter. Hence

1) The word "perihelion" is used in a generalised sense of that for the Newtonian mechanics.

2) Halphen, *loc. cit* p. 265.

this may be called a *quasi-elliptic* motion according to Whittaker.[1)] This type of motion corresponds to the ordinary Keplerian planetary motion. (Fig. 7a.) This advance of the perihelion corresponds to the secular motion of the perihelion in the ordinary theory of perturbation

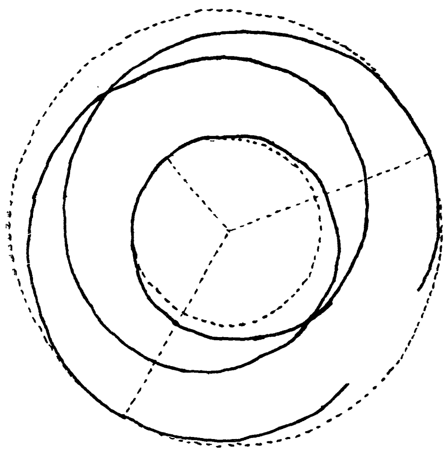

Fig. 7.
Quasi-Elliptic.
$2\pi < 2\omega < 3\pi$

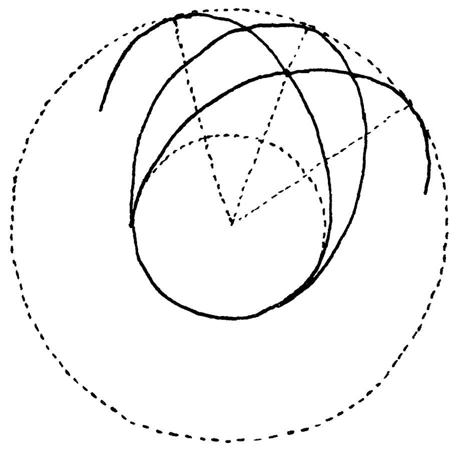

Fig. 7a.
Quasi-Elliptic.
$\pi < 2\omega < 2\pi$

in dynamical astronomy. In the Newtonian mechanics the radius vector r can not be represented by a simple harmonic motion, but the reciprocal of the radius vector is so expressed. In this relativistic mechanics neither r nor $\frac{1}{r}$ can be represented by a simple harmonic motion. The motion either of r or of $\frac{1}{r}$ is a simple harmonic motion, superposed by small oscillations with short periods of the aliquote parts of the principal periods, that is, superposed by a series of small oscillations proceeding in the harmonic terms of the arguments $\frac{n\pi}{\omega}(\psi' + \beta_1)$, $(n = 1, 2, 3, \ldots.)$. The coefficients of these successive terms, as we proceed further to the higher order harmonics decrease in magnitude, as $q < 1$.

If 2ω and π are in the ratio of an irrational number, then the ring region $r_2 \leqq r \leqq r_3$ is covered by the trace of the moving particle *everywhere*

1) Morton called this type of motion *general elliptic*, but I adopt Whittaker's terminology. The term has some relation to the word "quasi-periodicity" in the theory of algebraic functions. Cf., H. F. Baker, *An Introduction to the Theory of Multiply Periodic Functions* (1907).

densely.[1] This type of motion is called *quasi-ergodic*[2] in the recent terminology and it is *stable in the sense of Poisson* according to Poincaré.[3] Staude and later Charlier named this type of motion *conditionally periodic*[1] (*bedingt periodisch*), because, if that ratio in question is in the ratio of a rational number, the motion is strictly periodic. In this case the motion is said to be *degenerated* (*entartet*).

(Ib) $$e_3 < -\frac{1}{3} < e_2 < e_1 < \frac{2}{3}, \quad e_3 < x < e_2, \quad \mu > 1.$$

Put $$\psi = \psi' + 2\omega', \quad y = y' + \omega' \quad z = z, \quad h_3^2 > 1,$$
$$\beta_0 = \beta_0' + i\beta_0'', \quad \beta_3 = \beta_3' + i\beta_3'',$$

where β_0', β_0'', β_3' and β_3'' are real constants, then

$$\wp' y = +\frac{2\alpha}{h_1}\sqrt{h_3^2 - 1}, \quad \wp' z = -\frac{2\alpha h_3}{h_1}, \quad \wp'' y = \frac{2\alpha^2}{h_1^2}.$$

$$\begin{aligned}\frac{2(h_3^2-1)}{h_1}(s+\beta_0') = &\left\{\frac{1}{3} - \frac{\eta}{\omega} - \frac{2\alpha\pi}{\omega h_1\sqrt{h_3^2-1}}\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\sin\frac{n\pi}{\omega}y'\right\}(\psi'+\beta_1)\\ &-\frac{\pi}{\omega}\frac{\sin\frac{\pi}{2\omega}(\psi'+\beta_1)}{\cos\frac{\pi y'}{\omega} - \cos\frac{\pi}{2\omega}(\psi'+\beta_1)} + \frac{\alpha}{h_1\sqrt{h_3^2-1}}\log\frac{\sin\frac{\pi}{2\omega}\left(\frac{\psi'+\beta_1}{2}+y'\right)}{\sin\frac{\pi}{2\omega}\left(\frac{\psi'+\beta_1}{2}-y'\right)}\\ &-\sum_{n=1}^{\infty}\left\{\frac{4\pi}{\omega}\cos\frac{n\pi}{\omega}y' - \frac{4\alpha}{h_1\sqrt{h_3^2-1}}\frac{\sin\frac{n\pi}{\omega}y'}{n}\right\}\frac{q^{2n}}{1-q^{2n}}\sin\frac{n\pi}{2\omega}(\psi'+\beta_1),\end{aligned}$$

$$\begin{aligned}\frac{2}{\alpha h_3}(c_0 t + \beta_3') = &\left\{-\frac{1}{h_3}\left(\zeta z - \frac{\eta z}{\omega}\right) + \frac{\pi(2h_3^2-3)}{(h_3^2-1)^{3/2}\omega}\cdot\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\sin\frac{n\pi}{\omega}y'\right.\\ &\left. + \frac{h_1}{2\alpha(h_3^2-1)}\left(\frac{\eta}{\omega} - \frac{1}{3}\right)\right\}(\psi'+\beta_1)\\ &+\frac{h_1\pi}{2\alpha(h_3^2-1)\omega}\frac{\sin\frac{\pi}{2\omega}(\psi'+\beta_1)}{\cos\frac{\pi y'}{\omega} - \cos\frac{\pi}{2\omega}(\psi'+\beta_1)}\end{aligned}$$

1) Geiger u. Scheel, *Handbuch der Physik*. Bd. 5, Kap. IV by Fues; Charlier, *Mechanik des Himmels*. Bd. 1 (1902); Born, *Vorlesungen über Atommechanik*, Bd. 1 (1925).

2) Levi-Civita, Abhandlungen d. Hamburger Math. Seminar. Bd. 6 (1928) 323.

3) Poincaré, *Méthodes Nouvelles de la Mécanique Céleste*. T. 3 (1899); Journ. de Math. [iv] 1 (1885) 167; Hagihara, Japanese Journ. Astronomy and Geophysics. 5 (1927) 1. See, Chap. VIII of the present paper.

$$+\frac{2h_3^2-3}{2(h_3^2-1)^{3/2}}\log\frac{\sin\frac{\pi}{2\omega}\left(\frac{\psi'+\beta_1}{2}+y'\right)}{\sin\frac{\pi}{2\omega}\left(\frac{\psi'+\beta_1}{2}-y'\right)}$$

$$+\sum_{n=1}^{\infty}\left\{\frac{4}{h_3}\frac{\sin\frac{n\pi}{\omega}z}{n}+\frac{2q^n(2h_3^2-3)}{(h_3^2-1)^{3/2}}\sin\frac{n\pi}{\omega}y'\right.$$
$$\left.+\frac{2\pi h_1 q^n}{\alpha(h_3^2-1)\omega}\cos\frac{n\pi}{\omega}y'\right\}\frac{q^n}{1-q^{2n}}\sin\frac{n\pi}{2\omega}(\psi'+\beta_1),$$

$$u-\frac{1}{3}=-\frac{\eta}{\omega}-2\left(\frac{\pi}{\omega}\right)^2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}\cos\frac{n\pi}{2\omega}(\psi'+\beta_1),$$

$$-i\cdot\frac{2(h_3^2-1)}{h_1}\beta''_0=-2\eta'+\frac{2}{3}\omega'$$
$$+\frac{\alpha}{h_1\sqrt{h_3^2-1}}\left\{\frac{\pi i}{\omega}(\omega-y')-\frac{4\pi\omega'}{\omega}\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\sin\frac{n\pi}{\omega}y'\right\},$$

$$\frac{2i}{\alpha h_3}\beta_3''=-\frac{2}{h_3}(\omega'\zeta z-\eta' y)+\frac{h_1}{2\alpha(h_3^2-1)}\left(2\eta'-\frac{2}{3}\omega'\right)$$
$$-\frac{2h_3^2-3}{2(h_3^2-1)^{3/2}}\left\{\frac{\pi i}{\omega}(\omega-y')-\frac{4\pi\omega'}{\omega}\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\sin\frac{n\pi}{\omega}y'\right\}.$$

t becomes logarithmically infinite at (A). Two types of motion can be imagined.

(α) $-\frac{1}{3}<x\leqq e_2$.

(β) $e_3\leqq x<-\frac{1}{3}$. This is impossible because $r<0$.

The type (α) of the motion is asymptotic at $r=\infty$ and the domain is from $r_2=\dfrac{\alpha}{e_2+\frac{1}{3}}$ to $r=\infty$. The geometrical trajectory in space has two distinct asymptotes:

$$\psi=\overline{\psi}_{A2}\equiv-\beta_1-2y,$$

and

$$\psi=\overline{\psi}_{A1}\equiv-\beta_1+2y.$$

The moving particle approaches to the asymptote $\psi=\overline{\psi}_{A2}$ as $t\to-\infty$ if $h_3^2<\frac{3}{2}$, and as $t\to+\infty$ if $h_3^2>\frac{3}{2}$; and to the asymptote $\psi=\overline{\psi}_{A1}$ as $t\to-\infty$ if $h_3^2>\frac{3}{2}$, and as $t\to+\infty$ if $h_3^2<\frac{3}{2}$. The angle between the two asymptotes is $4y$. If $4y=2p\pi+\nu$, where p is a positive integer, positive or zero, and ν is a positive number less than 2π, then the particle performs p complete revolutions and a fraction of one revolution during those two asymptotic approaches. This type of motion can be called *quasi-hyperbolic.* (Fig. 8 and Fig. 8a.) As the point e_3 approaches to $-\frac{1}{3}$, the two asymptotes tend to coincide. If $e_3=-\frac{1}{3}$, the type of

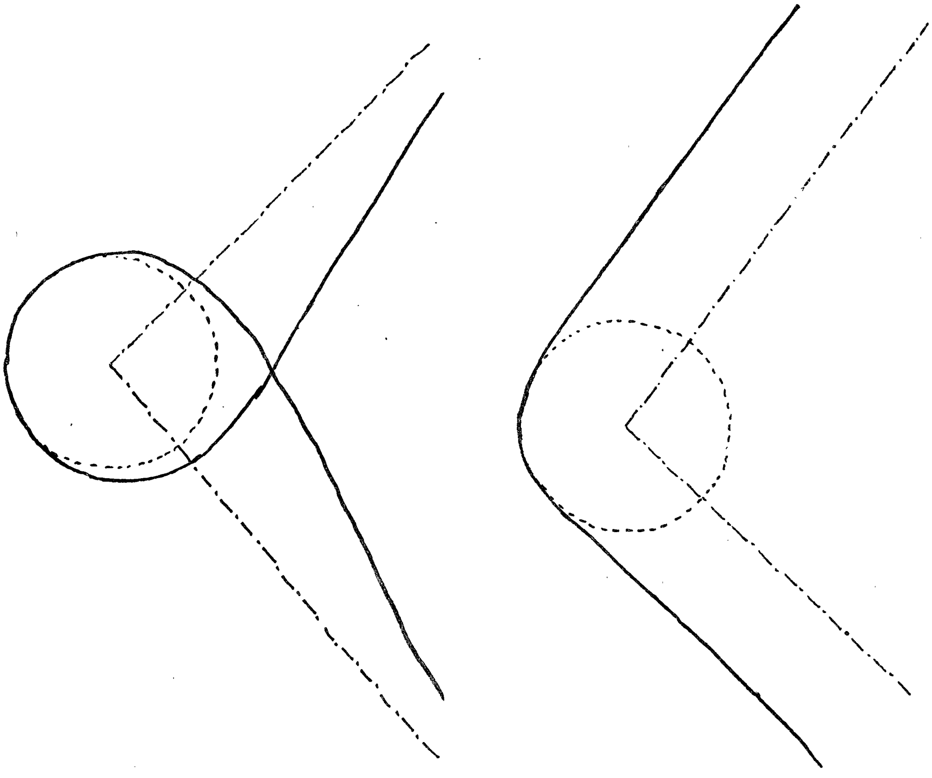

Fig. 8.
Quasi-Hyperbolic.
$2\pi<4y<3\pi$.

Fig. 8a.
Quasi-Hyperbolic.
$\pi<4y<2\pi$.

motion may be said to be *quasi-parabolic.* In the case of a quasi-parabolic motion, we have $y=\omega'$ and the axis of the quasi-parabola lies in $\psi+\beta_1=2\omega'$. As e_3 crosses $x=-\frac{1}{3}$, the type changes into the quasi-elliptic (IIa). These types of motion—quasi-elliptic, quasi-parabolic and quasi-hyperbolic—are considered to be a generalisation of the Keplerian types of motion in the Newtonian mechanics. If e_1 and e_2 coincide in the type (Ib), then the type (IIIa) comes in.

En résumé, the various types of motion ccorrsponding to the various cases of the distribution of the singularities are tabulated as follows:

Table V.

(Ia)	(α)	$0 \leqq r < \alpha$:	Inadmissible, Fig. 3;
	(β)	$\alpha < r \leqq \dfrac{\alpha}{e_1+\frac{1}{3}}$:	Pseudo-elliptic, Fig. 4.
(IIa)	(α)	$0 \leqq r < \alpha$:	Inadmissible, Fig. 3;
	(β)	$\alpha < r \leqq \dfrac{\alpha}{e_1+\frac{1}{3}}$:	Pseudo-elliptic, Fig. 4.
(IIIa)	(α)	$0 \leqq r < \alpha$:	Inadmissible, Fig. 5;
	(β)	$\alpha < r < \infty$:	Pseudo-hyperbolic, Fig. 6;
	(γ)		Impossible.
(IVa)	(α)	$0 \leqq r < \alpha$:	Inadmissible, Fig. 3;
	(β)	$\alpha < r \leqq \dfrac{\alpha}{e_k+\frac{1}{3}}$, ($k=1$ or 3):	Pseudo-elliptic, Fig. 4.
(Ib)	(α)	$\dfrac{\alpha}{e_2+\frac{1}{3}} \leqq r < \infty$;	Quasi-hyperbolic, Fig. 8 and Fig. 8a.
	(β)		Impossible.
(IIb)		$\dfrac{\alpha}{e_2+\frac{1}{3}} \leqq r \leqq \dfrac{\alpha}{e_3+\frac{1}{3}}$:	Quasi-elliptic, Fig. 7 and Fig. 7a.

The limits of r only with single inequality signs in this table are those of the asymptotic approaches as the time passes to infinity. The asymptotic character at $r=\alpha$ is not geometrically asymptotic. In the case of an asymptotic approach to infinity there exist two asymptotes for any geometrical trajectory. If these two asymptotes coincide, the motion is parabolic instead of hyperbolic, and such circumstances occur when $e_3 = -\dfrac{1}{3}$.

The general feature of these kinds of motion was studied by Charlier[1] in the general hyper-elliptic cases of the algebraic integrals[2]. He classified two types of motion: *libratory* and *limitatory*. Our type (IIb) belongs to the libratory motion of Charlier. The other types in this Chapter belong to the limitatory.

1) Charlier, *loc. cit.*

2) Appell et Goursat, *Théorie des Fonctions Algébraïques et de leurs Intégrales* (1895); H. F. Baker, *Abel's Theorem and the Allied Theory; An Introduction to the Theory of Multiply Periodic Functions.*

Chapter VIII.

Degenerate Cases.

Elliptic functions become *degenerated* when the discriminant of the fundamental cubic reduces to zero, *i. e.*, when one or both of the periods ω and $\frac{\omega'}{i}$ become infinite. The states of motion in such a case are discussed in this Chapter.

We have the following formulae from the theory of the elliptic functions:

(i) $\Delta=0$, $g_3>0$; then $e_2=e_3$, $\frac{\omega'}{i}=\infty$.

If we put $e_1=2a$, $(a>0)$, then $e_2=e_3=-a$,

$$g_2=12a^2, \quad g_3=8a^3.$$

$$-\frac{1}{2}e_1=e_2=e_3=-\frac{3g_3}{2g_2}, \quad \left(\frac{\pi}{2\omega}\right)^2=\frac{9g_3}{2g_2}=3a.$$

$$\wp w=-\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2+\left(\frac{\pi}{2\omega}\right)^2.\frac{2}{1-\cos\frac{\pi w}{\omega}},$$

$$\zeta w=\frac{\pi}{2\omega}\cot\frac{\pi w}{2\omega}+\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2 w,$$

$$\sigma w=e^{\frac{1}{6}\left(\frac{\pi w}{2\omega}\right)^2}.\frac{2\omega}{\pi}\sin\frac{\pi w}{2\omega},$$

$$\bar{\zeta} v=\frac{\pi}{2\omega}\coth\frac{\pi v}{2\omega}-\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2 v,$$

$$\eta\omega=\frac{\pi^2}{12},$$

$$\wp(v+\omega')=-\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2,$$

$$\zeta(v+\omega')=\frac{\eta v}{\omega}+\eta',$$

$$\log\sigma v=\frac{\eta v^2}{2\omega}.$$

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1) Appell et Lacour, *Principes de la Théorie des Fonctions Elliptiques et Applications.* p. 485; Halphen, *Traité des Fonctions Elliptiques et de leurs Applications.* T. 1 pp. 27, 90, 145, 183.

(ii) $\Delta=0$, $g_3<0$, then $e_1=e_2$, $\omega=\infty$.

If we put $e_1=e_2=a$, $(a>0)$, then $e_3=-2a$,

$$g_2=12a^2,\quad g_3=-8a^3,$$

$$-\frac{1}{2}e_3=e_1=e_2=-\frac{3g_3}{2g_2}.$$

$$\wp w=-\frac{2}{3}\left(\frac{\pi i}{2\omega'}\right)^2+\left(\frac{\pi i}{2\omega'}\right)^2\coth^2\frac{\pi wi}{2\omega'},$$

$$\zeta w=\frac{1}{3}\left(\frac{\pi i}{2\omega'}\right)^2 w+\left(\frac{\pi i}{2\omega'}\right)\coth\frac{\pi wi}{2\omega'},$$

$$\sigma w=\frac{2\omega'}{\pi i}\sinh\frac{\pi iw}{2\omega'}\cdot e^{-\frac{1}{6}\left(\frac{\pi iw}{2\omega'}\right)^2},$$

$$\zeta v=-\frac{1}{3}\left(\frac{\pi i}{2\omega'}\right)^2+\left(\frac{\pi i}{2\omega'}\right)\cot\frac{\pi vi}{2\omega'},$$

$$\eta'\omega'=\frac{\pi^2}{12},$$

$$\wp(v+\omega')=-\frac{2}{3}\left(\frac{\pi i}{2\omega'}\right)^2+\left(\frac{\pi i}{2\omega'}\right)^2\tanh^2\frac{\pi wi}{2\omega'},$$

$$\zeta(v+\omega')=\frac{1}{3}\left(\frac{\pi i}{2\omega'}\right)^2 v+\left(\frac{\pi i}{2\omega'}\right)\tanh\frac{\pi vi}{2\omega'},$$

$$\log\sigma_3 v=\frac{\eta v^2}{2\omega'}.$$

(iii) $\Delta=0$, $g_2=g_3=0$, then $e_1=e_2=e_3=0$.

$$\omega=\infty,\quad \frac{\omega'}{i}=\infty.$$

$$\wp w=\frac{1}{w^2},$$

$$\zeta w=\frac{1}{w},$$

$$\sigma w=w.$$

V. In this case $\Delta=0$, $g_2=g_3=0$ and $\omega=\infty$, $\dfrac{\omega'}{i}=\infty$. This is the only case in which the three roots of the cubic coincide at one point. It corresponds to the point of the cusp in Fig. 2. Further we have in this case

$$e_1=e_2=e_3=0,\quad \lambda=\frac{1}{3},\quad \mu=\frac{8}{9}.$$

$$y=i\sqrt{3},\quad z=\sqrt{\frac{3}{2}},\quad h_1^2=\frac{\alpha^2}{3},\quad h_3^2=\frac{80}{81}.$$

We distinguish two cases according to the domain of the motion.

(Va). The motion occurs in the domain: $e_1\leqq x\leqq\infty$.

$$\frac{2}{27\sqrt{3}\,\alpha}(s+\beta_0)=\frac{2}{3}(\psi+\beta_1)+\frac{4(\psi+\beta_1)}{(\psi+\beta_1)^2+12}-\sqrt{3}\tan^{-1}\left\{\frac{4\sqrt{3}(\psi+\beta_1)}{(\psi+\beta_1)^2-12}\right\},$$

$$\frac{\sqrt{3}}{4\sqrt{5}\,\alpha}(c_0t+\beta_3)=\frac{13}{4}(\psi+\beta_1)+\frac{3}{4\sqrt{6}}\log\frac{\psi+\beta_1+\sqrt{6}}{\psi+\beta_1-\sqrt{6}}$$
$$-11\sqrt{3}\tan^{-1}\left\{\frac{4\sqrt{3}(\psi+\beta_1)}{(\psi+\beta_1)^2-12}\right\}-9.\frac{\psi+\beta_1}{(\psi+\beta_1)^2+12},$$

$$u-\frac{1}{3}=\frac{4}{(\psi+\beta_1)^2}.$$

Two types of motion can be imagined.

(α) $0\leqq r<\alpha$: asymptotic and ejectional or asymptotic and collisional, but inadmissible by the reason often stated in the last chapter. Fig. 9a.

(β) $\alpha<r<3\alpha$: asymptotic on both sides. The trajectory approaches to the circle $r=3\alpha$ asymptotically, performing an infinite number of revolutions round the origin. Fig. 9b.

Consider (β) only. At $x=\frac{2}{3}$, *i. e.*, at $r=\alpha$, $\psi+\beta_1=\sqrt{6}$ or $\psi+\beta_1=-\sqrt{6}$ according as $t\to+\infty$ or $t\to-\infty$. Hence the trajectory is described from $r=\alpha$, $\psi+\beta_1=-\sqrt{6}$ to $r=3\alpha$ with indefinitely increasing ψ, or from $r=3\alpha$ with indefinitely decreasing ψ to $r=\alpha$, $\psi+\beta_1=\sqrt{6}$. As $r\to3\alpha$, the angular velocity tends to

$$\frac{d\psi}{dt}\to\frac{\sqrt{3}\,c_0}{13\sqrt{5}\,\alpha},$$

but never reduces to zero. Thus the rotation continues indefinitely as the time passes to infinity. On the other hand as $r\to\alpha$, the angular velocity tends to zero in such a way that

$$\frac{d\psi}{dt}\to\sqrt{\frac{2}{5}}\frac{c_0}{\alpha}e^{\sqrt{\frac{2}{5}}\frac{1}{\alpha}(c_0t+\beta_3)}\quad\text{as}\quad t\to-\infty,$$

and

$$\frac{d\psi}{dt}\to-\sqrt{\frac{2}{5}}\frac{c_0}{\alpha}e^{-\sqrt{\frac{2}{5}}\frac{1}{\alpha}(c_0t+\beta_3)}\quad\text{as}\quad t\to+\infty.$$

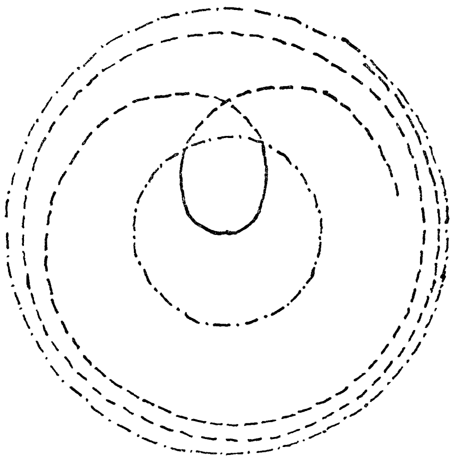

Fig. 9 a.
Inadmissible.

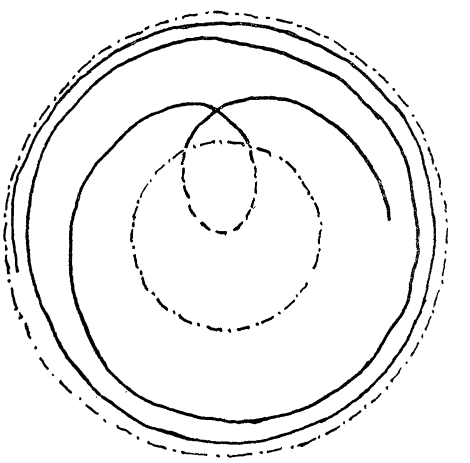

Fig. 9 b.
Pseudo-Spiral.

As was explained in the last Chapter this point $r=\alpha$ is the place where the spatial co-ordinates become standing still. Fig. 9b.

(Vb). The motion occurs in the domain: $e_3 \leqq x \leqq e_2$. But as $e_2 = e_3$ the domain for x shrinks to a mere point $e_1 = e_2 = e_3 = 0 = x$.

$$\frac{1}{9\sqrt{3}\,\alpha}(s+\beta_0) = \psi + \beta_1,$$

$$\frac{\sqrt{3}}{\sqrt{5}\,\alpha}(c_0 t + \beta_3) = 13(\psi + \beta_1),$$

$$u - \frac{1}{3} = 0.$$

This type of motion is circular with the radius $r=3\alpha$. The mean motion of the angular variable is $\dfrac{\sqrt{3}\,c_0}{13\sqrt{5}\,\alpha}$.

This circle is the limiting circle to which the type of motion (Va) tends as $t \to -\infty$ and as $t \to +\infty$, and is itself the trajectory of a possible motion. The initial constants of the motion should exactly have the numerical value corresponding to this type of motion. If it differs by the slightest amount, then the type of motion assumes the form (Va). There is only one definite possible value for the radius of the circular motion, while in the other cases such as (VIb) the radius of the circular motion can have any value out of a continuous stretch just like in the Keplerian circular motion in the Newtonian mechanics. Hence we distinguish this type of circular motion from the other types and name it a *pseudo-circular* motion.

The characteristic feature of this asymptotic approach to $r=3\alpha$ is that the asymptotic behaviour is algebraic, while in the other cases it is transcendental as will be shown later. Fig. 9c.

VI. This case of the distribution of the singularities corresponds to the lower boundary curve in Fig. 2 between the region for the type (II) and the region for the type (IV). According to the domain of the motion we distinguish two cases as usual.

(VI a)

$$\frac{2(h_3^2-1)}{h_1}(s+\beta_0)=\left\{\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2-\frac{1}{3}-\frac{\alpha}{h_1\sqrt{1-h_3^2}}\frac{\pi}{2\omega}\coth\frac{\pi y'}{2\omega}\right\}(\psi+\beta_1)$$

$$-\frac{\pi}{\omega}\frac{\sin\dfrac{\pi}{2\omega}(\psi+\beta_1)}{\cosh\dfrac{\pi y'}{\omega}-\cos\dfrac{\pi}{2\omega}(\psi+\beta_1)}$$

$$-\frac{\alpha}{h_1\sqrt{1-h_3^2}}\tan^{-1}\left\{\frac{\sinh\dfrac{\pi}{\omega}y'\sin\dfrac{\pi}{2\omega}(\psi+\beta_1)}{1-\cosh\dfrac{\pi}{\omega}y'\cos\dfrac{\pi}{2\omega}(\psi+\beta_1)}\right\},$$

$$\frac{2}{\alpha h_3}(c_0t+\beta_3)=\left\{-\frac{1}{h_3}\frac{\pi}{2\omega}\cot\frac{\pi z}{2\omega}-\frac{1}{\sqrt{1-h_3^2}}\frac{2h_3^2-3}{2(h_3^2-1)}\frac{\pi}{2\omega}\coth\frac{\pi y'}{2\omega}\right.$$
$$\left.+\frac{h_1}{2\alpha(h_3^2-1)}\left(\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2-\frac{1}{3}\right)\right\}(\psi+\beta_1)$$

$$+\frac{1}{h_3}\log\frac{\sin\dfrac{\pi}{2\omega}\left(\dfrac{\psi+\beta_1}{2}+z\right)}{\sin\dfrac{\pi}{2\omega}\left(\dfrac{\psi+\beta_1}{2}-z\right)}$$

$$+\frac{h_1\pi}{2\alpha(h_3^2-1)\omega}\cdot\frac{\sin\dfrac{\pi}{2\omega}(\psi+\beta_1)}{\cosh\dfrac{\pi}{\omega}y'-\cos\dfrac{\pi}{2\omega}(\psi+\beta_1)}$$

$$-\frac{1}{\sqrt{1-h_3^2}}\frac{2h_3^2-3}{2(h_3^2-1)}\tan^{-1}\left\{\frac{\sinh\dfrac{\pi}{\omega}y'\sin\dfrac{\pi}{2\omega}(\psi+\beta_1)}{1-\cosh\dfrac{\pi}{\omega}y'\cos\dfrac{\pi}{2\omega}(\psi+\beta_1)}\right\},$$

$$u-\frac{1}{3}=-\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2+\left(\frac{\pi}{2\omega}\right)^2\cdot\frac{2}{1-\cos\frac{\pi}{2\omega}(\psi+\beta_1)},$$

where

$$\left(\frac{\pi}{2\omega}\right)^2=3a.$$

The geometrical trajectory reaches its greatest distance from the origin for $\psi+\beta_1=\frac{\pi}{\sqrt{3a}}$ and comes to the origin for $\psi+\beta_1=0$ or $\frac{2\pi}{\sqrt{3a}}$. This geometrical trajectory is divided by the circle $r=\alpha$ into two parts.

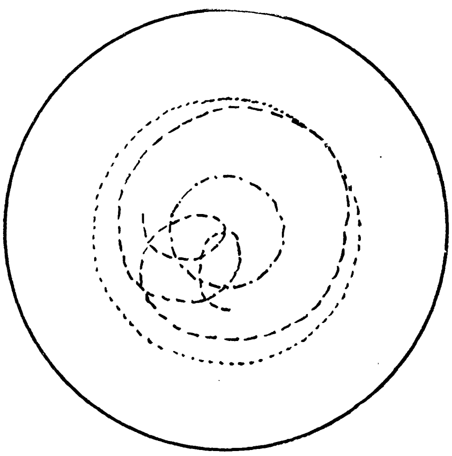

Fig. 9 c.
Pseudo-Circular.

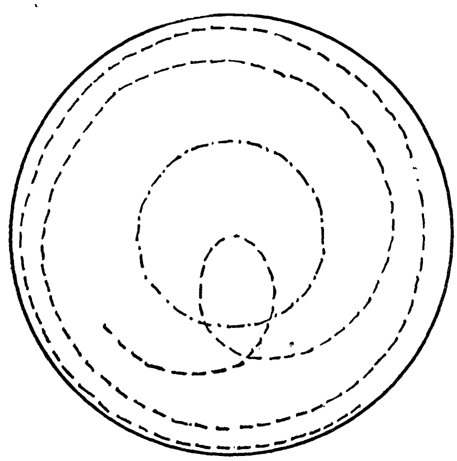

Fig. 10.
Circular.

The moving particle can not cross over this circle. The azimuthal angles of the points of intersection of $r=\alpha$ with the geometrical trajectory are $\psi+\beta_1=\pm 2z$. The moving particle approaches asymptotically to $r=\alpha$, $\psi+\beta_1=2z$ as $t\to-\infty$ and to $r=\alpha$, $\psi+\beta_1=-2z$ as $t\to+\infty$. The general character of the motion is similar to that in (IIa), but the small oscillations of short periods in that type die out as the two roots e_2 and e_3 tend to coincide. This type is the linkage between (IIa) and (IVa).

(α) $-\frac{1}{3}<-a<2a<\frac{2}{3}<x$: Inadmissible. Fig. 3.

(β) $-\frac{1}{3}<-a<2a<x<\frac{2}{3}$: Pseudo-elliptic. Fig. 4.

(VI b)

$$\frac{2(h_3^2-1)}{h_1}(s+\beta_0')=\left\{\frac{1}{3}-\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2-\frac{\alpha}{h_1\sqrt{1-h_3^2}}\frac{\pi}{2\omega}\coth\frac{\pi y'}{2\omega}\right\}(\psi'+\beta_1),$$

$$\frac{2}{\alpha h_3}(c_0 t+\beta_3')=\left\{-\frac{1}{h_3}\frac{\pi}{2\omega}\cot\frac{\pi z}{2\omega}-\frac{1}{\sqrt{1-h_3^2}}\frac{2h_3^2-3}{2(h_3^2-1)}\frac{\pi}{2\omega}\coth\frac{\pi y'}{2\omega}\right.$$

$$\left.+\frac{h_1}{2\alpha(h_3^2-1)}\left(\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2-\frac{1}{3}\right)\right\}(\psi'+\beta_1),$$

$$u=e_2+\frac{1}{3}=e_3+\frac{1}{3}=\frac{1}{3}-a,$$

with

$$\beta_0''=-\infty,\qquad \beta_3''=-\infty,$$

$$\left(\frac{\pi}{2\omega}\right)^2=3a.$$

This is an ordinary circular motion. Fig. 10. The mean motion in the argument of latitude ψ' is $\frac{c_0}{mh_3}$ divided by the factor of $(\psi'+\beta_1)$ in the second equation. This is the degenerate case of the type (IIb) and it corresponds to the Keplerian circular motion. This case is the only linkage between the relativistic and the Newtonian trajectories. As the two roots e_2 and e_3 separate, a quasi-elliptic motion with small eccentricity, so to speak, will appear and gradually changes into (IIb), if we adopt the relativistic mechanics; and an elliptic motion appears, if we adopt the Newtonian.

VII. $$-\frac{1}{3}<e_1<e_2=e_3>\frac{2}{3}.$$

This corresponds to the upper boundary curve between the region for the cases (II) and the region for (IV). This case of the distribution is the linkage between the latter two cases when the two roots e_2 and e_3 of the fundamental cubic tend to coincide. As was already stated, we have

$$e_1=e_2=a,\qquad e_3=-2a,$$

$$\left(\frac{\pi i}{2\omega'}\right)^2=3a.$$

We distinguish two sub-cases:

(VII a) $$e_1<x<+\infty.$$

$$\frac{2(h_3^2-1)}{h_1}(s+\beta_0)=\left\{\frac{1}{3}-\frac{1}{3}\left(\frac{\pi i}{2\omega'}\right)^2-\frac{\alpha}{h_1\sqrt{1-h_3^2}}\frac{\pi i}{2\omega'}\cot\frac{\pi i y'}{2\omega'}\right\}(\psi+\beta_1)$$

$$-\frac{\pi i}{\omega'}\frac{\sinh\frac{\pi i}{2\omega'}(\psi+\beta_1)}{\cos\frac{\pi i y'}{\omega'}-\cosh\frac{\pi i}{2\omega'}(\psi+\beta_1)}$$

$$-\frac{\alpha}{h_1\sqrt{1-h_3^2}}\tan^{-1}\left\{\frac{\sin\frac{\pi i}{\omega'}y'\sinh\frac{\pi i}{2\omega'}(\psi+\beta_1)}{1-\cos\frac{\pi i}{\omega'}y'\cosh\frac{\pi i}{2\omega'}(\psi+\beta_1)}\right\},$$

$$\frac{2}{\alpha h_3}(c_0t+\beta_3)=\left\{-\frac{1}{h_3}\frac{\pi i}{2\omega'}\coth\frac{\pi i z}{2\omega'}-\frac{1}{\sqrt{1-h_3^2}}\frac{2h_3^2-3}{2(h_3^2-1)}\frac{\pi i}{2\omega'}\cot\frac{\pi i}{2\omega'}y'\right.$$
$$\left.+\frac{h_1}{2\alpha(h_3^2-1)}\left(\frac{1}{3}\left(\frac{\pi i}{2\omega'}\right)^2-\frac{1}{3}\right)\right\}(\psi+\beta_1)$$

$$+\frac{h_1\pi i}{2\alpha(h_3^2-1)\omega'}\cdot\frac{\sinh\frac{\pi i}{2\omega'}(\psi+\beta_1)}{\cos\frac{\pi i y'}{\omega'}-\cosh\frac{\pi i}{2\omega'}(\psi+\beta_1)}$$

$$+\frac{1}{h_3}\log\frac{\sinh\frac{\pi i}{2\omega'}\left(\frac{\psi+\beta_1}{2}+z\right)}{\sinh\frac{\pi i}{2\omega'}\left(\frac{\psi+\beta_1}{2}-z\right)}$$

$$-\frac{1}{\sqrt{1-h_3^2}}\frac{2h_3^2-3}{2(h_3^2-1)}\tan^{-1}\left\{\frac{\sin\frac{\pi i}{\omega'}y'\sinh\frac{\pi i}{2\omega'}(\psi+\beta_1)}{1-\cos\frac{\pi i}{\omega'}y'\cosh\frac{\pi i}{2\omega'}(\psi+\beta_1)}\right\},$$

$$u-\frac{1}{3}=-\frac{2}{3}\left(\frac{\pi i}{2\omega'}\right)^2+\left(\frac{\pi i}{2\omega'}\right)^2\coth^2\frac{\pi i}{4\omega'}(\psi+\beta_1).$$

The motion is devoid of small oscillations of short periods which appeared in the non-degenerate cases. The principal feature is the asymptotic approach to the circle $r=\dfrac{\alpha}{\frac{1}{3}+a}$ with an infinite number of revolutions round the origin. In fact, $\coth^2\frac{\pi i}{4\omega'}(\psi+\beta_1)\rightarrow+1$ as $\psi\rightarrow\pm\infty$. Hence the limiting value of $u-\frac{1}{3}$ is $a=e_1=e_2=\frac{1}{3}\left(\frac{\pi i}{2\omega'}\right)^2$. Thus the moving particle approaches doubly asymptotically to the circle $r=\dfrac{\alpha}{\frac{1}{3}+a}$

describing a spiral orbit both as $t \to +\infty$ and as $t \to -\infty$. The geometrical trajectory of this case is an example of *doubly asymptotic orbits* of Poincaré,[1] if we consider the central body as a mathematical point. Actually it is ejectional and asymptotic. But the dynamical trajectory is cut by the circle $r=\alpha$ and the moving particle can not cross over this barrier. At $r=\alpha$ the geometrical trajectory is not asymptotic. The motion is asymptotic to $r=\alpha$, $\psi+\beta_1=-2z$ for $t\to+\infty$, and to $r=\alpha$, $\psi+\beta_1=2z$ for $t\to-\infty$. This circle $r=\alpha$ divides the types of motion into two:

$$(\alpha) \quad \frac{2}{3}<x<\infty: \qquad \text{Inadmissible.}$$

This motion, if existed, would be either ejectional from $r=0$ and asymptotic to $r=\alpha$, or asymptotic from $r=\alpha$ and collisional to $r=0$. Fig. 9a.

$$(\beta) \quad e_1<x<\frac{2}{3}:$$

The asymptotic approach to $x=\frac{2}{3}$, *i. e.*, $r=\alpha$, $\psi+\beta_1=\pm 2z$, so occurs that the velocity tends to zero, but the asymptotic approach to $x=e_1=a$ occurs in such a way that the trajectory, both dynamical and geometrical, winds round and round describing a spiral orbit as it approaches to the circle with a non-vanishing angular velocity. However this asymptotic approach to $r=\dfrac{\alpha}{\frac{1}{3}+a}$ in this case is different from that in the case (Va β), although the appearance is very similar. The approach in (VIIa β) is exponential, while that in (Va β) is algebraic, that is, x admits $\psi=\infty$ as a transcendental singularity in (VIIa β) but in (Va β) it is a double pole. Hence the type of motion of this case (VIIa β) may be called *transcendentally pseudo-spiral*, while the type in (Va β) *algebraically pseudo-spiral*. In spite of this difference we demonstrate it by the same figure. Fig. 9.

(VII b) $\qquad e_3<x<e_2.$

$$\frac{2(h_3^2-1)}{h_1}(s+\beta_0')=\left\{\frac{1}{3}-\frac{1}{3}\left(\frac{\pi i}{2\omega'}\right)^2-\frac{\alpha}{h_1\sqrt{1-h_3^2}}\frac{\pi i}{2\omega'}\cot\frac{\pi i y'}{2\omega'}\right\}(\psi+\beta_1),$$

$$\frac{2}{\alpha h_3}(c_0t+\beta_3')=\left\{-\frac{1}{h_3}\frac{\pi i}{2\omega'}\coth\frac{\pi i z}{2\omega'}-\frac{1}{\sqrt{1-h_3^2}}\frac{2h_3^2-3}{2(h_3^2-1)}\frac{\pi i}{2\omega'}\cot\frac{\pi i y'}{2\omega'}\right.$$

1) H. Poincaré, *Méthodes Nouvelles de la Mécanique Céleste*, T. 3.

$$+\frac{h_1}{2\alpha(h_3^2-1)}\left(\frac{1}{3}\left(\frac{\pi i}{2\omega'}\right)^2-\frac{1}{3}\right)\Bigg\}(\psi+\beta_1),$$

$$u-\frac{1}{3} \qquad =-\frac{2}{3}\left(\frac{\pi i}{2\omega'}\right)^2+\left(\frac{\pi i}{2\omega'}\right)^2\tanh^2\frac{\pi i}{4\omega'}(\psi+\beta_1),$$

$$\beta_0''=-\infty, \qquad \beta_3''=-\infty.$$

The type of motion is one of the most interesting in the relativistic mechanics. The motion is doubly asymptotic to the same circle $r=\dfrac{\alpha}{\frac{1}{3}+a}$ performing an infinite number of revolutions round the origin in the form of a spiral, both as $t\to+\infty$ and as $t\to-\infty$. This is a typical example of Poincaré's *doubly asymptotic orbits*[1]. In fact, as $\psi+\beta_1=0$, then $u-\dfrac{1}{3}=-\dfrac{2}{3}\left(\dfrac{\pi i}{2\omega'}\right)^2$, *i. e.*, $r=\dfrac{\alpha}{\frac{1}{3}-2a}$. As $\psi+\beta_1\to\pm\infty$, then $u-\dfrac{1}{3}\to$ $\dfrac{1}{3}\left(\dfrac{\pi i}{2\omega'}\right)^2$, *i. e.*, $r\to\dfrac{\alpha}{\frac{1}{3}+a}$. Obviously the value of $\dfrac{d\psi}{dt}$ taken from the second equation does not vanish at $t\to\pm\infty$. Hence the motion starts from $r=\dfrac{\alpha}{\frac{1}{3}+a}$ asymptotically for $t\to-\infty$ and reaches the maximum distance $r=\dfrac{\alpha}{\frac{1}{3}-2a}$ from the origin at $t=-\dfrac{\beta_3'}{c_0}$ and approaches back to $r=\dfrac{\alpha}{\frac{1}{3}+a}$ asymptotically for $t\to+\infty$, performing an infinite number of revolutions round the origin. Fig. 11.

Another interesting feature of this type of motion is that the circle $r=\dfrac{\alpha}{\frac{1}{3}+a}$ is a typical example of the *cycles limites* of Poincaré. The two types of motion (VII a β) and (VII b) both approach to this circle asymptotically from both sides, each performing an infinite number of revolutions round the centre. This is what I have mentioned in the case (II). Fig. 11a.

1) H. Poincaré, *Méthodes Nouvelles de la Mécanique Céleste.* T. **3** (1899) Chap. XXXIII.

2) H. Poincaré, Journn. de Math. [iii] **8** (1882) 251.

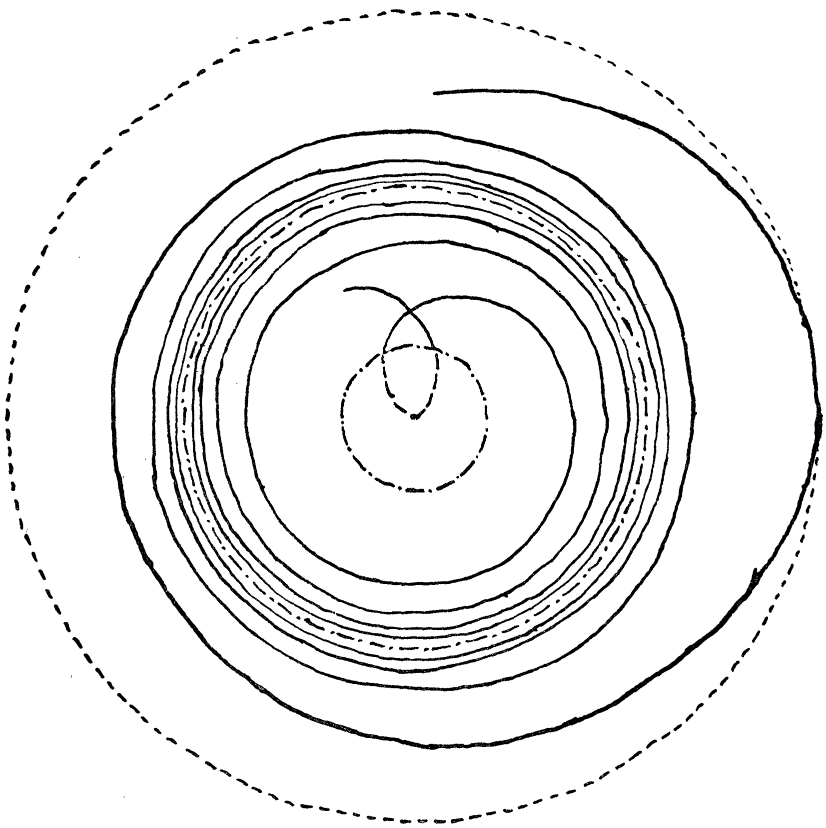

Fig. 11 a.
Cycle Limite.

These two types of motion, which tend to the cycle limite from both sides, are each consisted of an infinite number of individual motions corresponding to an infinite number of values of the constants of integration in the continuous stretches of their domain of possible values. Hence a swarm of particles must be found scattering over on both sides of the circumference of this circle $r=\dfrac{\alpha}{\dfrac{1}{3}+a}$, in a similar manner found in Saturn's ring or in the asteroidal ring in our Solar System. But, as $a>0$, the radius of this circle lies between α and 3α, and is of the order of magnitude of α. Thus it is improbable for such an interesting specimen of dynamics actually to exist in Nature by the reason often quoted in Chapter VI. Also in the type of motion (V) there ought to be a swarm of particles along the circumference of the circle $r=3\alpha$ in

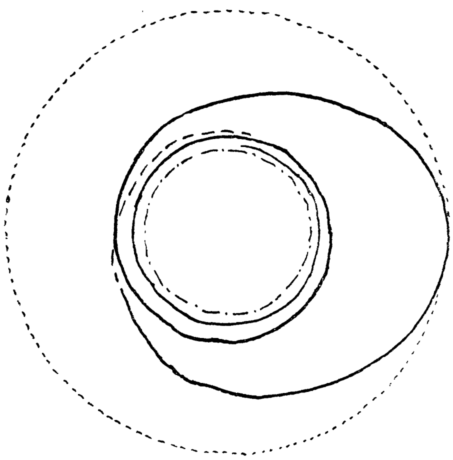

Fig. 11.
Quasi-Elliptic Spiral.
(Doubly Asymptotic).

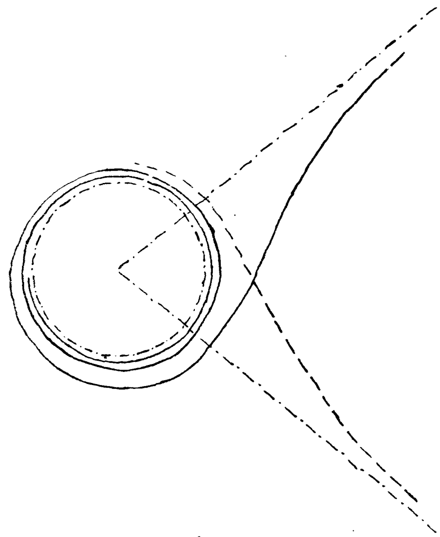

Fig. 12.
Quasi-Hyperbolic Spiral.

a similar way but only on one side $r<3\alpha$. This is also physically improbable by the same reason.

The type of motion (VII b) may be called *quasi-elliptic spiral.* Especially, if $e_3=-2a=-\frac{1}{3}$, then the type of motion is asymptotic at the point $r=\infty$ and may be called *quasi-parabolic spiral.* In this case $e_1=e_2=a=\frac{1}{6}$.

VIII. $$e_3<-\frac{1}{3}<e_2=e_1<\frac{2}{3}.$$

This case is the linkage between the cases (I) and (III) and corresponds to the boundary curve in Fig. 2 between the region for the case (I) and the region for (III).

(VIII a). $$e_1<x<\infty.$$

$$\frac{2(h_3^2-1)}{h_1}(s+\beta_0)=\left\{\frac{1}{3}-\frac{1}{3}\left(\frac{\pi i}{2\omega'}\right)^2-\frac{\alpha}{h_1\sqrt{h_3^2-1}}\left(\frac{\pi i}{2\omega'}\right)\coth\frac{\pi i}{2\omega'}y\right\}(\psi+\beta_1)$$

$$-\frac{\pi i}{\omega'}\frac{\sinh\frac{\pi i}{2\omega'}(\psi+\beta_1)}{\cosh\frac{\pi i}{\omega'}y'-\cosh\frac{\pi i}{2\omega'}(\psi+\beta_1)}$$

$$+\frac{\alpha}{h_1\sqrt{h_3^2-1}}\log\frac{\sinh\frac{\pi i}{2\omega'}\left(\frac{\psi+\beta_1}{2}+y\right)}{\sinh\frac{\pi i}{2\omega'}\left(\frac{\psi+\beta_1}{2}-y\right)},$$

$$\frac{2}{\alpha h_3}(c_0t+\beta_3)=\left\{-\frac{1}{h_3}\left(\frac{\pi i}{2\omega'}\right)\coth\frac{\pi i}{2\omega'}z+\frac{1}{\sqrt{h_3^2-1}}\frac{2h_3^2-3}{2(h_3^2-1)}\frac{\pi i}{2\omega'}\coth\frac{\pi i}{2\omega'}y\right.$$

$$\left.+\frac{h_1}{2\alpha(h_3^2-1)}\left(\frac{1}{3}\left(\frac{\pi i}{2\omega'}\right)^2-\frac{1}{3}\right)\right\}(\psi+\beta_1)$$

$$+\frac{h_1\pi}{2\alpha(h_3^2-1)}\frac{i}{\omega'}\cdot\frac{\sinh\frac{\pi i}{2\omega'}(\psi+\beta_1)}{\cosh\frac{\pi i}{\omega'}y-\cosh\frac{\pi i}{2\omega'}(\psi+\beta_1)}$$

$$+\frac{1}{h_3}\log\frac{\sinh\frac{\pi i}{2\omega'}\left(\frac{\psi+\beta_1}{2}+z\right)}{\sinh\frac{\pi i}{2\omega'}\left(\frac{\psi+\beta_1}{2}-z\right)}$$

$$-\frac{1}{\sqrt{h_3^2-1}}\frac{2h_3^2-3}{2(h_3^2-1)}\log\frac{\sinh\frac{\pi i}{2\omega'}\left(\frac{\psi+\beta_1}{2}+y\right)}{\sinh\frac{\pi i}{2\omega'}\left(\frac{\psi+\beta_1}{2}-y\right)},$$

$$u-\frac{1}{3}=-\frac{2}{3}\left(\frac{\pi i}{2\omega'}\right)^2+\left(\frac{\pi i}{2\omega'}\right)^2\coth^2\frac{\pi i}{4\omega'}(\psi+\beta_1).$$

The type of motion is asymptotic at $x=a=e_1=e_2$, *i. e.*, $r=\dfrac{\alpha}{\frac{1}{3}+a}$, performing an infinite number of revolutions round the origin. Also it is asymptotic, not geometrically but only dynamically, to the point of intersection of the geometrical trajectory with the circle $r=\alpha$. The general feature of the motion is quite similar to that of (VII a).

There are two types of motion which can be imagined.

(α) $0<r<\alpha$: Inadmissible. Fig. 9a.

(β) $\alpha<r<\dfrac{\alpha}{\frac{1}{3}+a}$: Transcendentally pseudo-spiral. Fig. 9b.

(VIIIb) $e_3<x<e_2$.

$$\frac{2(h_3^2-1)}{h_1}(s+\beta_0')=\left\{\frac{1}{3}-\frac{1}{3}\left(\frac{\pi i}{2\omega'}\right)^2-\frac{\alpha}{h_1\sqrt{h_3^2-1}}\left(\frac{\pi i}{2\omega'}\right)\coth\frac{\pi i}{2\omega'}y'\right\}(\psi'+\beta_1)$$

$$-\frac{\pi i}{\omega'}\frac{\sinh\frac{\pi i}{2\omega'}(\psi'+\beta_1)}{\cosh\frac{\pi i}{\omega'}y'-\cosh\frac{\pi i}{2\omega'}(\psi'+\beta_1)}$$

$$+\frac{\alpha}{h_1\sqrt{h_3^2-1}}\log\frac{\sin\frac{\pi i}{2\omega'}\left(\frac{\psi'+\beta_1}{2}+y'\right)}{\sin\frac{\pi i}{2\omega'}\left(\frac{\psi'+\beta_1}{2}-y'\right)},$$

$$\frac{2}{\alpha h_3}(c_0t+\beta'_3)=\left\{-\frac{1}{h_3}\left(\frac{\pi i}{2\omega'}\right)\coth\frac{\pi i z}{2\omega'}+\frac{1}{\sqrt{h_3^2-1}}\frac{2h_3^2-3}{2(h_3^2-1)}\frac{\pi i}{2\omega'}\coth\frac{\pi i}{2\omega'}y'\right.$$

$$\left.+\frac{h_1}{2\alpha(h_3^2-1)}\left(\frac{1}{3}\left(\frac{\pi i}{2\omega'}\right)^2-\frac{1}{3}\right)\right\}(\psi'+\beta_1)$$

$$+\frac{h_1\pi}{2\alpha(h_3^2-1)}\frac{i}{\omega'}\frac{\sinh\frac{\pi i}{2\omega'}(\psi'+\beta_1)}{\cosh\frac{\pi i y'}{\omega'}-\cosh\frac{\pi i}{2\omega'}(\psi'+\beta_1)}$$

$$+\frac{1}{\sqrt{h_3^2-1}}\frac{2h_3^2-3}{2(h_3^2-1)}\log\frac{\sinh\frac{\pi i}{2\omega'}\left(\frac{\psi'+\beta'}{2}+y'\right)}{\sinh\frac{\pi i}{2\omega'}\left(\frac{\psi'+\beta_1}{2}-y'\right)},$$

$$u-\frac{1}{3}=-\frac{2}{3}\left(\frac{\pi i}{2\omega'}\right)^2+\left(\frac{\pi i}{2\omega'}\right)^2\tanh^2\frac{\pi i}{4\omega'}(\psi'+\beta_1),$$

$$\beta_0''=-\infty,\quad \beta_3''=-\infty.$$

The type of motion at $r=\dfrac{\alpha}{\frac{1}{3}+a}$ is quite similar to that of (VIIb). But this type of motion extends to infinity, and indeed has the character of an asymptotic approach to $r=\infty$. The asymptotes to which the motion tends at infinity are determined by $\psi'+\beta_1'=\pm 2y'$. If $2h_3^2-3>0$, the moving particle tends to $\psi'+\beta_1'=2y'$ as $t\to+\infty$ and to $\psi'+\beta_1'=-2y'$ as $t\to-\infty$. If $2h_3^2-3<0$, it tends to $\psi'+\beta_1'=2y'$ as $t\to-\infty$ and to $\psi'+\beta_1'=-2y'$ as $t\to+\infty$. The motion either starts from one of the asymptotes from infinity and approaches to the circle $r=\dfrac{\alpha}{\frac{1}{3}+a}$, performing an infinite number of revolutions as the time tends to infinity,

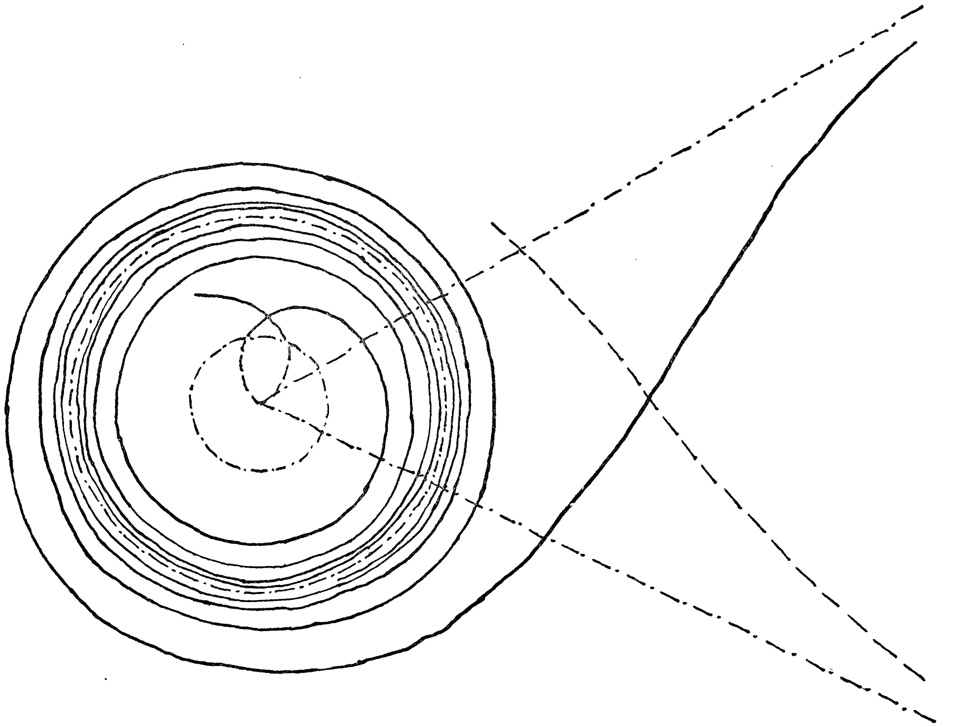

Fig. 12a. Cycle Limite.

or starts from the circle $r=\dfrac{\alpha}{\dfrac{1}{3}+a}$, performing an infinite number of revolutions from an infinitely long past and approaches to one of the asymptotes indefinitely to infinity as the time passes to infinity. Fig. 12.

The circle $r=\dfrac{\alpha}{\dfrac{1}{3}+a}$ is a *cycle limite* of Poincaré. Fig. 12a.

This type of motion may be called *quasi-hyperbolic spiral*. Especially if e_3 coincides with $-\dfrac{1}{3}$, then the motion may be said to be *quasi-parabolic spiral*. Thus the type of motion may be changed into quasi-elliptic spiral (VIIb) as e_3 crosses over $x=-\dfrac{1}{3}$ towards $x=0$.

Chapter VIII.

Correlation between Various Types of Motion.

How are the various types of motion correlated to each other? How do they vary as the constants of integration change continuously?

In order to answer this question we reproduce our Fig. 2 here and enlarge it in Fig. 13 regardless of the scale of measure. Various parts of the domain are named as shown in Fig. 13. (I) (II) (III) (IV) (V) (VI) (VII) (VIII) are those already treated in Chap. VI and Chap. VII.

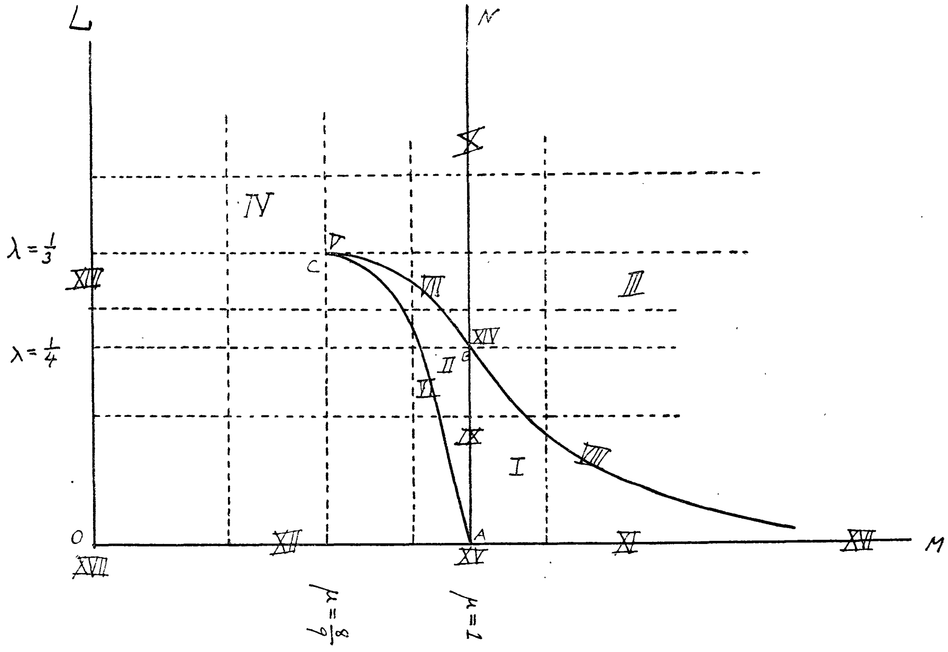

Fig. 13.

(IX) $\mu=1, \qquad 0<\lambda<\frac{1}{4}.$

Boundary between (I) and (II) and e_3 has a special value $-\frac{1}{3}$, that is, $y=\omega'$.

(IXa): Pseudo-elliptic,

(IXb): Quasi-parabolic, as already treated in (V).

(X) $\mu=1, \qquad \frac{1}{4}<\lambda<\infty.$

Boundary between (III) and (IV) and e_3 has a special value $-\frac{1}{3}$.
Pseudo-parabolic.

(XI) $\lambda=0, \qquad 1<\mu<\infty.$

$$e_1=\frac{2}{3}, \qquad e_2=e_3=-\frac{1}{3}, \qquad \Delta=0.$$

This may be considered as the limiting case of (I). The real domains (Ia) and (Ib) shrink to mere points, $\frac{2}{3}$ and $-\frac{1}{3}$, respectively.

As $\frac{\omega'}{i}=\infty$ in this case, $\pi=2\omega$, $z=\frac{\pi}{2}$, $y=i\infty$, $e_1=\frac{2}{3}=\wp z=\frac{2}{3}\left(\frac{\pi}{2\omega}\right)^2$.
The type of motion (XIa) is a standing still at $r=\alpha$, $\psi+\beta_1=\pi$, as $\frac{d\psi}{dt}=0$.
The type of motion (XIb) is also a standing still at $r=\infty$, $\psi+\beta_1=$ any value.

(XII) $\lambda=0, \qquad 0<\mu<1.$

The circumstance is similar to (XI).

(XIII) $\mu=0, \qquad 0<\lambda<\infty. \qquad e_1=\frac{2}{3}.$

e_2 and e_3 are complex and can have any values.
This is the limting case of (IV) in which e_1 tends to $\frac{2}{3}$. In the type of motion corresponding to the upper part in the right hand corner of the domain (IV) it is proper to consider the only real root to be e_1. Hence the type of motion is a standing still at $r=\alpha$, $\psi+\beta_1=\pi$.

(XIV) $\lambda=\frac{1}{4}, \qquad \mu=1.$

This type of motion is the junction of four types (I) (II) (III) and (IV). This may be considered as the limiting case of (VII) in which e_3 tends to $-\frac{1}{3}$. The type of motion (XIVb) may be called *quasi-parabolic spiral*, while the type (XIVaβ) is pseudo-spiral just like (VIIaβ). The type (XIVaα) is inadmissible. The circle $r=\dfrac{\alpha}{e_1+\frac{1}{3}}=\dfrac{\alpha}{e_2+\frac{1}{3}}$ is a *cycle limite*. This case is the linkage between (VII) and (VIII), and also the linkage between (IX) and (X).

(XV) $\lambda=0$, $\mu=1$.

This is the limit of (VI) in which $e_2=e_3$ tends to $-\frac{1}{3}$ and e_1 tends to $\frac{2}{3}$ at the same time. The nature of the motion is similar to (XI).

(XVI) $\lambda=0$, $\mu=\infty$.

This is the limit of (XI). The limit of (VIII) is $e_1=e_2=\frac{1}{3}$, $e_3=-\frac{2}{3}$. There is always a domain (I) between these two types (XI) and

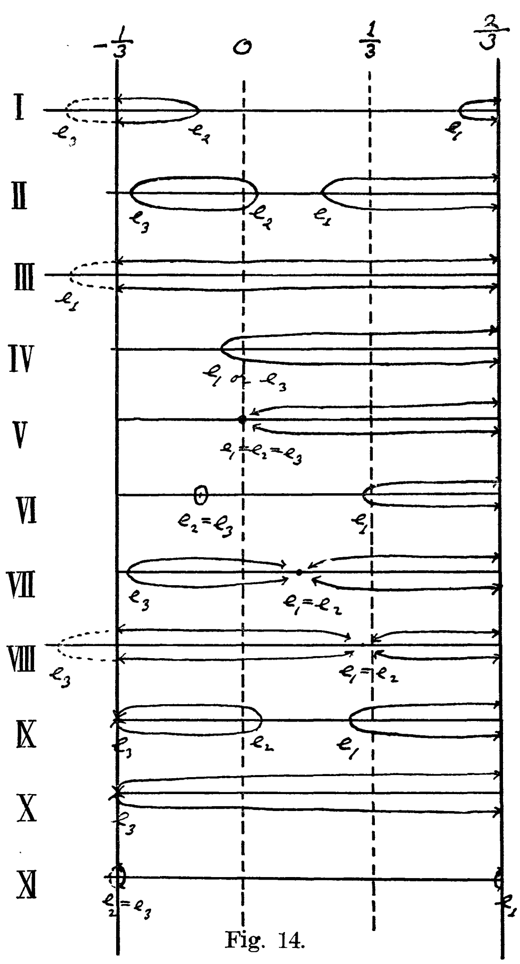

Fig. 14.

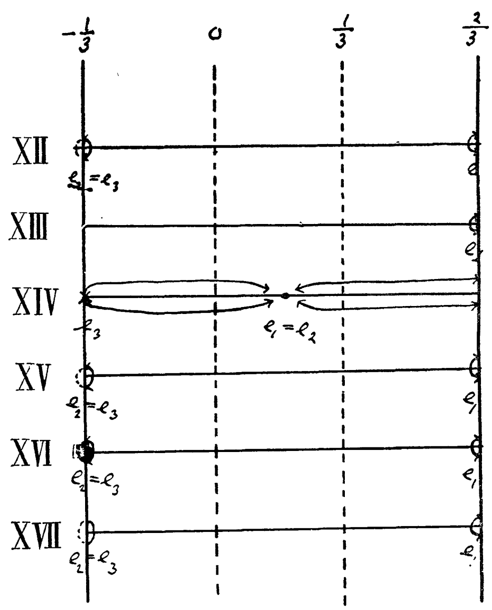

Fig. 14. conti.

(VIII) even at $\mu \to \infty$.

(XVII) $\lambda=0$, $\mu=0$.

The circumstance is similar to (XII).

A rough idea of the real domains and of the types of motion in the various cases can be obtained from Fig. 14.

The correlation between these various types of motion can be best sought for in Fig. 13. If we draw a curve showing the manner in which we vary the constants of integration, then the types of motion corresponding to the various domains of the figure through which the curve passes are those which we are seeking. I think that there is hardly any need of explaining it in detail. The use of Fig. 14 together with Fig. 13 is quite sufficient for that purpose.

But what is the physical meaning of the constants of integration λ and μ which we so often refer to?

By our construction

$$\lambda=\frac{\alpha^2}{h_1{}^2}, \qquad \mu=h_3{}^2, \quad \text{and} \quad \alpha=2m,$$

where m denotes the mass of the central body at the origin measured in the unit of length. If α increases from zero, λ increases. For large values of λ the type of motion is either (IV) or (III), that is, pseudo-hyperbolic, or pseudo-parabolic, or pseudo-elliptic. (See Fig. 13 or Fig. 2.) It can be imagined that the large mass would attract the particle so immensely that it soon comes to the nearest approach to the centre and tends to stand still on the circle $r=\alpha$. This is what the actual circumstance is. u contains α. But it is only to change the scale of measure and the type of motion for any value of α can be obtained by a similarity transformation from the type of motion for another value of α.

As to the physical meaning of h_3 and h_1, it is not so simple as to attribute any concrete idea to them, except in the case of a quasi-elliptic planetary orbit. (Chap. X.)

Suppose that α is kept constant. Then the same relative correlation between various types of motion, as that for increasing values of α with constant h_1, is also valid for decreasing values of h_1 with constant α. The values of α are related to the absolute position of the singularities of the elliptic integrals. Thus α and h_1 are mixed together in exhibiting themselves in the physical nature of our problem. For very small values of α or for very large values of h_1, the type of motion approaches either to the type (XI), or to (XII), or to (XV), or to (XVI), or to (XVII). These are the cases in which $\frac{\omega'}{i}$ is infinitely great and $\frac{\pi}{2\omega}$ is nearly equal to unity. In fact, as was shown in the case (IIb) in Chap. VI by referring to the formulae:

$$\sqrt{\frac{2\omega}{\pi}}\sqrt[4]{e_1-e_2}=1-2q^2+2q^4-\cdots,$$

$$\sqrt{\frac{2\omega}{\pi}}\sqrt[4]{e_1-e_3}=1+2q^2+2q^4+\cdots,$$

$\frac{2\omega}{\pi}$ tends to unity, as e_1-e_2 or e_1-e_3 tends to unity. But e_1 ought to lie between 0 and $\frac{2}{3}$. Hence e_1-e_3 or e_1-e_2 can tend to unity only if e_2 and e_3 both tend to $-\frac{1}{3}$ and at the same time e_1 tends to $\frac{2}{3}$. This case corresponds to the above mentioned five types. Hence $\frac{\pi}{2\omega}$ is equal to unity for those types of motion.

In order to find out the physical meaning of the constants of integration h_3, we form the Newtonian kinetic energy of the moving particle at infinity, when it is actually or virtually brought to an infinite distance from the origin of the co-ordinates. As the motion occurs on one plane during the whole time, the Newtonian kinetic energy is $\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\psi}{dt}\right)^2$. By transforming this we get

$$\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\psi}{dt}\right)^2=\left[\left(\frac{d\psi}{du}\right)^2\frac{\alpha^2}{u^2}+\frac{\alpha^4}{u^4}\right]\left(\frac{du}{dt}\right)^2.$$

From (49) this is equal to

$$\frac{h_1^2c_0^2}{h_3^2}(1-u)^2U(u)\left[u^5-u^4+\frac{\alpha^2}{h_1^2}u^3+\frac{\alpha^2(h_3^2-1)}{h_1^2}u^2+\alpha^2\right].$$

At an infinite distance from the origin $u\to 0$, and hence the Newtonian kinetic energy at infinity is $\frac{\alpha^4(h_3^2-1)c_0^2}{h_3^2}$. At $h_3^2=\infty$ this expression has the value 1. As h_3^2 decreases from infinity to 1, this expression decreases from 1 to 0, and then, as h_3^2 decreases to zero, the Newtonian kinetic energy at infinity, keeping its negative value, decreases to $-\infty$.

We have put $\mu=h_3^2$. $\mu=1$ divides the types of motion from the elliptic to the hyperbolic, both for the quasi- and the pseudo- types. Hence, if the Newtonian kinetic energy at infinity is positive, then the types of motion are hyperbolic; and if it is negative, then the types are elliptic. Particularly, if it is zero, then the types of motion are parabolic, and if it is negatively infinite, then the types of motion are those for the light rays, as will be shown in Chap. IX. This circumstance beautifully corresponds to the Keplerian motion in the Newtonian mechanics.

It is very remarkable that the actually existing planets in the Solar System have their trajectories of the type (IIb) but very near to (XV), corresponding to the parts for small values of α in Fig. 13.

This circumstance shows that for the actually existing planets the kinetic energy at infinity is negative, and, besides, the mass of the Sun is very small. It is vaguely imagined that these two facts correspond to a nearly stable end-state from an analogous consideration to the one in the statistical mechanics. But it is not a place to talk on such a cosmological problem here.

Chapter IX.

TRAJECTORY OF A LIGHT RAY.

In Chap. III we have obtained the result that for the trajectory of a light ray we should take $K=0$ in the equation (89):

$$U(u)=u^3-u^2+\frac{2K\alpha^2}{h_1^2}u+\frac{\alpha^2(h_3^2-2K)}{h_1^2};$$

that is, if we denote the expression $U(u)$ for a light ray by $U_0(u)$, we should consider the equation:

$$U_0(u)=u^3-u^2+\frac{\alpha^2h_3^2}{h_1^2}.$$

For the trajectory of a moving particle this expression was reduced to

$$U(u)=u^3-u^2+\frac{\alpha^2}{h_1^2}u+\frac{\alpha^2(h_3^2-1)}{h_1^2}.$$

In Chap. V we have transformed this into:

$$U(u)=u^3-u^2+\lambda u-\lambda(1-\mu),$$

by putting

$$\lambda=\frac{\alpha^2}{h_1^2},\quad \mu=h_3^2.$$

The assumption that $K=0$ is equivalent to taking

$$\lambda=0,\quad \mu=\infty,$$

but such that

$$\lambda\mu=\frac{\alpha^2 h_3^{\,2}}{h_1^{\,2}},$$

in our discussion for the trajectories of massless particles.

Let us turn to Fig. 13. The domain $\lambda=0$, $\mu=\infty$ is for the type (III) or (I) or (VIII) or (XVI). In order to decide to which domain the type of motion of a light ray belongs, we examine the behaviour of the curve $\lambda\mu=\frac{\alpha^2 h_3^{\,2}}{h_1^{\,2}}$, relative to the boundary curve $\Delta=0$. The branch in question of the boundary curve is given by

$$\mu=\frac{2}{27\lambda}\left\{(9\lambda+1)+\sqrt{-(3\lambda-1)^3}\right\}.$$

For the curve $\lambda\mu=\frac{\alpha^2 h_3^{\,2}}{h_1^{\,2}}$, the ordinate μ' is given by

$$\mu'=\frac{\alpha^2 h_3^{\,2}}{\lambda h_1^{\,2}}.$$

As
$$\mu-\mu'=\frac{2}{27\lambda}\left\{(9\lambda+1)+\sqrt{-(3\lambda-1)^3}\right\}-\frac{1}{\lambda}\cdot\alpha^2\frac{h_3^{\,2}}{h_1^{\,2}},$$

behaves in the limit $\lambda\to 0$ as

$$\frac{4}{27\lambda}-\frac{1}{\lambda}\alpha^2\frac{h_3^{\,2}}{h_1^{\,2}},$$

so, according as

$$\frac{4}{27}-\frac{\alpha^2 h_3^{\,2}}{h_1^{\,2}}\lesseqqgtr 0,$$

the curve $\lambda\mu=\frac{\alpha^2 h_3^{\,2}}{h_1^{\,2}}$ is situated above, or on, or below the boundary curve $\Delta=0$. Hence the boundary curve $\Delta=0$ is in the domain (III) or (VIII) or (I), according as

$$\frac{4}{27}\lesseqqgtr\frac{\alpha^2 h_3^{\,2}}{h_1^{\,2}}.$$

The trajectory of a light ray, as it lies on this curve $\lambda\mu=\frac{\alpha^2 h_3^{\,2}}{h_1^{\,2}}$ in the part $\lambda\to 0$, is of the type (III) or (VIII) or (I), according as

$$\frac{4}{27}\lesseqqgtr\frac{\alpha^2 h_3^{\,2}}{h_1^{\,2}}.$$

In this case we have the following relations:

$$g_2=\frac{4}{3},\quad g_3=4\left(\frac{2}{27}-\frac{\alpha^2 h_3^{\,2}}{h_1^{\,2}}\right),$$

$$\wp'^2 y = \frac{4\alpha^2 h_3^2}{h_1^2}, \quad \wp'' y = 0,$$

$$\wp'^2 z = \frac{4\alpha^2 h_3^2}{h_1^2} = \wp'^2 y.$$

To take $\lambda = 0$ is equivalent to assuming that $h_1 = 0$. Hence we get $s + \beta_0 = 0$ in (68). Thus the proper time or the co-ordinate time is always zero. This result is quite in accord with the principle of relativity.

Corresponding to the type of motion (Ia) there is a trajectory, which, starting from a point with the radius vector $r = \alpha$, goes back to another point with the same value of the radius vector, without receding farther than $r = \frac{3\alpha}{2}$ from the center. If the radius of the central star is $\alpha < r < \frac{3}{2}\alpha$, then some light rays emitted by the atoms on the surface of the star can not entirely leave its neighbourhood but strikes the surface again in a finite interval of time. Even if the type is (VIIIa), the circle of its asymptotic approach is of the radius $2\alpha < r < 3\alpha$. Hence for a physically existing star it is highly improbable by the reason already given in Chap. VI, that an observer lying outside a very massive star can not see the star at all because of the impossibility of the escape of light rays from its neighbourhood, a fact often cited in some of the treatises on relativity. Thus the light rays emitted by the atoms on the surface of a star always reaches to infinity and there exist always some light rays coming from infinity which strike the surface of any star. Hence, if we neglect the cosmological term in the general theory of relativity as is assumed in this paper, then we have the result that an observer in any part of the space can see every star which is duly thought to be existing by physical consideration on the possible density of the stars, however massive the star may be.

As in the motion of a particle, a light ray comes to a standing still on the circle $r = \alpha$. But this is practically improbable by the same reason as in the case of the motion of a particle.

Chapter X.

Quasi-Elliptic Motion.

1. Let us confine our attention to the important case (IIb), which contains the ordinary planetary motions.

By (52) and (56) we have

$$\left.\begin{aligned} dt&=\frac{\alpha^2 h_3}{h_1 c_0}\frac{dx}{\left(\frac{2}{3}-x\right)\left(x+\frac{1}{3}\right)^2\sqrt{(x-e_1)(x-e_2)(x-e_3)}}, \\ 0&=\frac{dx}{\sqrt{(x-e_1)(x-e_2)(x-e_3)}}-d\psi. \end{aligned}\right\} \tag{80}$$

Put

$$\left.\begin{aligned} \frac{dx}{\sqrt{(x-e_1)(x-e_2)(x-c_3)}}&=\frac{\omega}{\pi}dw_1, \quad d\psi=dw_2, \\ x=q_1, &\qquad \psi=q_2, \end{aligned}\right\}$$

as a slightly more general form than that of Charlier[1)] and further put

$$\left.\begin{aligned} dt&=F_{11}(q_1)dw_1+F_{12}(q_2)dw_2, \\ 0&=F_{21}(q_1)dw_1+F_{22}(q_2)dw_2, \end{aligned}\right\}$$

where

$$\left.\begin{aligned} F_{11}&=\frac{\alpha^2 h_3}{h_1 c_0}\frac{1}{\left(\frac{2}{3}-q_1\right)\left(q_1+\frac{1}{3}\right)^2}\frac{\omega}{\pi}, \quad F_{12}=0, \\ F_{21}&=\frac{\omega}{\pi}, \qquad F_{22}=-1. \end{aligned}\right\}$$

The matrix of half-periods:

$$\omega_{ij}=\int_0^\pi F_{ij}(q_i)dw_i, \quad (i,j=1,2)$$

is calculated to be

$$\begin{pmatrix} \omega_{11} & 0 \\ \omega_{21} & \omega_{22} \end{pmatrix},$$

with

$$\omega_{11}=\frac{2\alpha^2 h_3\omega}{h_1 c_0}\left\{-\frac{1}{h_3}\left(\zeta z-\frac{\eta z}{\omega}\right)-\frac{1}{\sqrt{1-h_3^2}}\frac{2h_3^2-3}{2(h_3^2-1)}\left(\zeta y'+\frac{\eta}{\omega}y'\right)\right\}=\Omega, \text{ say},$$

$$\omega_{21}=2\omega,$$

$$\omega_{22}=-\pi.$$

1) Charlier, *Mechanik des Himmels.* Bd. 1.

Construct
$$\left.\begin{aligned}\pi\left(t+\frac{\beta_3}{c_0}\right)&=\omega_{11}l_1+\omega_{12}l_2,\\ \mu\beta_1&=\omega_{21}l_1+\omega_{22}l_2\,;\end{aligned}\right\}$$
or
$$l_1=\frac{\pi}{\Omega}\left(t+\frac{\beta_3}{c_0}\right),\quad l_2=\frac{2\omega}{\Omega}\left(t+\frac{\beta_3}{c_0}\right);$$
or
$$l_1=n_1t+\varpi_1,\quad l_2=n_2t+\varpi_2,$$
with
$$\left.\begin{aligned}n_1&=\frac{\pi}{\Omega},\qquad n_2=\frac{2\omega}{\Omega},\\ \varpi_1&=\frac{\pi\beta_3}{\Omega c_0},\qquad \varpi_2=\frac{2\omega\beta_3}{\Omega c_0}+\beta_1.\end{aligned}\right\}\tag{81}$$

Ω depends on h_1 and h_3 as arbitrary constants.

Then the solution q_1 and q_2 can be written in the form:
$$\left.\begin{aligned}q_j&=\sum_{\nu_1=-\infty}^{+\infty}\sum_{\nu_2=-\infty}^{+\infty}C^{(j)}_{\nu_1\nu_2}e^{(\nu_1l_1+\nu_2l_2)\sqrt{-1}},\quad (j=1,2)\\ \text{with}\quad \pi^2C^{(j)}_{\nu_1\nu_2}&=\int_0^{\pi}\int_0^{\pi}q_j(l_1,l_2)e^{-(\nu_1l_1+\nu_2l_2)\sqrt{-1}}dl_1dl_2.\end{aligned}\right\}\tag{82}$$
$$(j=1,2\,;\ \nu_1,\nu_2=-\infty,\ldots,+\infty)$$

Further by (45) and (48) we obtain the values of the co-ordinates:

r: radius vector,
ψ: argument of latitude,
φ: co-latitude,
θ: longitude,

by the formulae:
$$\left.\begin{aligned}&\frac{2m}{r}-\frac{1}{3}=q_1,\\ &\psi=q_2,\\ &\cos\varphi=\sin I\sin\psi,\\ &\tan(\theta+\beta_2)=\tan\psi\cos I.\end{aligned}\right\}\tag{83}$$

The arbitrary constants are $h_1, h_2, h_3, \beta_1, \beta_2, \beta_3$.

$C^{(j)}_{\nu_1\nu_2}$'s $(j=1,2\,;\nu_1,\nu_2=-\infty,\ldots,+\infty)$ depend on h_1 and h_3. h_1, h_3, β_1 and β_3 determine the form of the orbit, while h_2 and β_2 determine the position of the plane of the orbit on the plane of reference.

The expression (82) are in the form often employed in atomic physics. If ν_1 and ν_2 are connected by a linear homogeneous relation with rational coefficients, then the motion is said to be *degenerated* (*entartet*).[1] Otherwise the representative point of the motion in the plane of r and ψ covers the manifold $l_1 l_2$ *everywhere densely*. This case is called *quasi-ergodic*.[2] Thus generally the periods of r and of ψ do not coincide and hence the motion of the perihelion occurs. The perihelion advances by the amount:

$$(4\omega-2\pi)\frac{4\omega}{2\pi}=4\omega\left(\frac{2\omega}{\pi}-1\right)$$

in an anomalistic revolution, that is, for one complete period of r.

2. As was proved already, we have $\frac{2\omega}{\pi}>1$ in this case. Hence the point of the maximum radius vector and the minimum radius vector, or in short, the aphelion and the perihelion, advance in the sense of the motion of the particle. If $4\omega=2p\pi+\nu$, where p is an integer and ν is a constant less than 2π, then the perihelion performs p revolutions and a fraction of one revolution in one period of r, that is, in a time interval between subsequent two epochs in which the radius vector passes its initial value in the same sense. ν is the apparent advance of the perihelion. If $\pi<\nu<2\pi$, then it looks as though the perihelion were in a retrograde motion. As e_2 and e_3 approach to each other, the orbit tends to be circular as was shown in Chap. VII.

From the theory of elliptic functions we get[3]

$$\left.\begin{aligned}
&\left(\frac{2\omega}{\pi}\right)^2 e_1-\frac{2}{3}=16\sum\frac{nq^{2n}}{1-q^{2n}},\\
&\left(\frac{2\omega}{\pi}\right)^2 e_2+\frac{1}{3}=-16\sum\frac{p(-q)^p}{1+(-q)^p},\\
&\left(\frac{2\omega}{\pi}\right)^2 e_3+\frac{1}{3}=-16\sum\frac{pq^p}{1-q^p},
\end{aligned}\right\}\quad\left(\begin{aligned}&n=1,3,5,\ldots\\&p=1,2,3,\ldots\end{aligned}\right)\tag{84}$$

in which $$q=e^{-\frac{\pi\omega'}{\omega i}}.$$

As $\frac{\omega'}{i}\to\infty$, then $q\to 0$ and

$$e_1\to\frac{2}{3}\left(\frac{\pi}{2\omega}\right)^2,$$
$$e_2\to-\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2,$$

1) Geiger u. Scheel, *Handbuch der Physik*. Bd. 5 Kap. IV; Charlier, *Mechanik des Himmels*. Bd. 1; Born, *Vorlesungen über Atommechanik*.

2) T. Levi-Civita, Abhandlungen Hamburg Math. Seminar. Bd. 6 (1928) 323.

3) Halphen, *loc. cit.* p. 447.

$$e_3 \rightarrow -\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2.$$

The amount of the advance of the perihelion keeps the same value as we vary $\frac{\omega'}{i}$ into infinity and it has the same value also in the limit $\frac{\omega'}{i} \rightarrow \infty$. It is remarked that $\frac{e_2}{e_1}$ and $\frac{e_3}{e_1}$ both tend to $-\frac{1}{2}$, whatever the value of $\frac{\pi}{2\omega}$ may be.

Now the solution for the relativistic trajectories has four constants of integration, namely, h_1, h_3, β_1, β_3, or λ, μ, β_1, β_3. If we restrict our attention only to the form of the trajectories, then β_1 and β_3 do not enter into our discussion, for they are only to shift the zero-point of the longitude and that of the time. λ and μ enter in the fundamental cubic equation which has played an important rôle in the above investigation; that is, in the equation:

$$u^3 - u^2 + \lambda u - \lambda(1-\mu) = 0,$$

or

$$4x^3 - g_2 x - g_3 = 0$$

where

$$g_2 = 4\left(\frac{1}{3} - \lambda\right), \quad g_3 = 4\left(\frac{2}{27} + \frac{2}{3}\lambda - \lambda\mu\right),$$

$$e_1 + e_2 + e_3 = 0, \quad e_1 e_2 + e_2 e_3 + e_3 e_1 = -\left(\frac{1}{3} - \lambda\right),$$

$$e_1 e_2 e_3 = \frac{2}{27} + \frac{2}{3}\lambda - \lambda\mu,$$

$$\Delta = -64\lambda\left[\lambda^2 + \left(2 - 9\mu + \frac{27}{4}\mu^2\right)\lambda + (1-\mu)\right].$$

We can transform our independent arbitrary constants λ and μ into ω and $\frac{\omega'}{i}$, connected by the relations:

$$\wp\omega = e_1, \quad \wp\omega' = e_3.$$

Or, by putting

$$l = \frac{\sqrt[4]{e_1 - e_3} - \sqrt[4]{e_1 - e_2}}{\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_1 - e_2}}$$

$$h = \frac{1}{2}l + 2\left(\frac{1}{2}l\right)^5 + 15\left(\frac{1}{2}l\right)^9 + \dots,$$

the relations take the form:[1]

1) Weierstrass-Schwarz, *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen* (1893) 61.

$$\left.\begin{aligned} &\sqrt{\frac{2\omega}{\pi}}=\frac{2}{\sqrt[4]{e_1-e_3}+\sqrt[4]{e_1-e_2}}(1+2h^4+2h^{16}+\ldots\ldots), \\ &\frac{\omega'}{i}=\frac{\omega}{\pi}\log\left(\frac{1}{h}\right), \\ &\sqrt{\frac{2\omega}{\pi}}\sqrt[8]{\frac{\Delta}{2}}=\frac{\pi}{2\omega}h^{\frac{1}{4}}(1-3h^2+5h-\ldots\ldots). \end{aligned}\right\} \tag{85}$$

These are rather complicated and it is hard to write down any simple formulae for this transformation. Especially the condition for a double root $\Delta=0$ is quite beyond our control.[1)]

3. Suppose that in this way we have changed the independent arbitrary constants of integration from λ, μ to $\omega, \frac{\omega'}{i}$. From an ∞^2 number of these solutions, we choose out an ∞^1 of them such that $\frac{\omega'}{i}=\infty$. These latter solutions correspond to the circular trajectories.

Now the radius of the circular orbits are computed by

$$\mathbf{a}_0=\frac{\alpha}{e_2+\frac{1}{3}}=\frac{3\alpha}{1-\left(\frac{\pi}{2\omega}\right)^2}. \tag{86}$$

α is twice the mass of the central body measured iu the unit of length. From our discussion in the case (VIb) Chap. VII, we get the sidereal period of these circular motions to be

$$T_0=\frac{\pi\alpha}{c_0}\left[\frac{\pi}{2\omega}\cot\frac{\pi z}{2\omega}-\frac{\wp' z}{\wp' y}\left(1-\frac{\wp'' y}{\wp'^2 y}\right)\frac{\pi}{2\omega}\cos\frac{\pi y}{2\omega}-\frac{\wp' z}{3\wp'^2 y}\left(1-\left(\frac{\pi}{2\omega}\right)^2\right)\right].$$

$\wp$, $\wp'$ and $\wp''$ contain ω. y and z are determined by the relations:

$$\wp z=\frac{2}{3}, \qquad \wp y=-\frac{1}{3}.$$

Hence by the formulae in the beginning of Chap. VII, we get

$$\wp' y=-\frac{2i}{3\sqrt{3}}\left\{1-\left(\frac{\pi}{2\omega}\right)^2\right\}\sqrt{1+2\left(\frac{\pi}{2\omega}\right)^2},$$

$$\wp' z=-\frac{2}{3\sqrt{3}}\left\{2+\left(\frac{\pi}{2\omega}\right)^2\right\}\sqrt{2-2\left(\frac{\pi}{2\omega}\right)^2},$$

1) The relation may be obtained from

$$\left.\begin{aligned} &\left(\frac{2\omega}{\pi}\right)^4\frac{g_2}{20}=\frac{1}{15}+\frac{4}{1.2}\sum\sum(2n)^3q^{nl}, \\ &\left(\frac{2\omega}{\pi}\right)^6\frac{g_3}{28}=\frac{2}{3^3.7}-\frac{4}{1.2.3.4.}\sum\sum(2n)^5q^{2nl}. \end{aligned}\right\} \quad n, l=1, 2, 3, \ldots\ldots$$

Still it is complicated and I omit to copy them here. Cf., Halphen, *loc. cit.* p. 446.

$$\wp''y=\frac{2}{3}\left\{1+\left(\frac{\pi}{2\omega}\right)^2\right\}\left\{1-\left(\frac{\pi}{2\omega}\right)^2\right\},$$

$$\cot\frac{\pi z}{2\omega}=\sqrt{\frac{2}{3}}\cdot\frac{\sqrt{1-\left(\frac{\pi}{2\omega}\right)^2}}{\left(\frac{\pi}{2\omega}\right)},$$

$$\cot\frac{\pi y}{2\omega}=-\frac{i}{\sqrt{3}}\frac{\sqrt{1+2\left(\frac{\pi}{2\omega}\right)^2}}{\left(\frac{\pi}{2\omega}\right)}.$$

Thus

$$T_0=\frac{6\sqrt{6}\pi\alpha}{c_0}\cdot\frac{1}{\left\{1-\left(\frac{\pi}{2\omega}\right)^2\right\}^{\frac{3}{2}}}. \tag{87}$$

Hence if we substitute from (86)

$$\left(\frac{\pi}{2\omega}\right)^2=1-\frac{3\alpha}{\mathbf{a}_0}$$

in this expression, we get a relation containing $\mathbf{a}_0$ and T_0 with α or m:

$$T_0^2=\frac{8\pi^2\mathbf{a}_0^3}{c_0^2\alpha}.$$

This is the generalised Kepler's third law.

Suppose that

$$\left(\frac{\pi}{2\omega}\right)^2=1-\gamma, \tag{88}$$

where γ is a small quantity of the order of magnitude $\frac{3\alpha}{\mathbf{a}_0}$, then

$$T_0=\frac{2\pi\sqrt{6}\mathbf{a}_0}{c_0\sqrt{\gamma}}+\ldots\ldots$$

An approximate value of γ can be obtained from this equation. Thus

$$\gamma=\frac{24\pi^2\mathbf{a}_0^2}{c_0^2T_0^2}.$$

Now the motion of the perihelion is

$$4\omega\left(\frac{2\omega}{\pi}-1\right)\approx 2\pi\left(\frac{1}{\sqrt{1-\gamma}}-1\right)\approx\pi\gamma,$$

in one anomalistic revolution. By inserting the above value for γ we get an approximate value of the advance of the perihelion to be

$$\frac{24\pi^3 \mathbf{a}_0^2}{c_0^2 T_0^2}, \tag{89}$$

which is in good agreement with Einstein's formula, when the eccentricity of the orbit is neglected.

More generally, if (86) is substituted in (87), we get

$$T_0 = \frac{2\pi\sqrt{6}\,\mathbf{a}_0}{c_0} \cdot \frac{1}{\left\{1-\left(\frac{\pi}{2\omega}\right)^2\right\}^{\frac{3}{2}}}. \tag{90}$$

This is the generalised formula of (89) and gives the advance of the perihelion $4\omega\left\{\frac{2\omega}{\pi}-1\right\}$.

Supposing that there is a planet close to the surface of the companion of Sirius, its period of revolution is computed to be several minutes and the advance of its perihelion to be a few seconds of arc. The perihelion performs a complete revolution in not less than a century. This is the case of the greatest deviation of the relativistic mechanics from the Newtonian that may be realised in the physical world. Hence these formulae of ours are unfortunately quite of little use for practical purpose.

This type of circular motion exists in the interval $-\frac{1}{3}<x<0$, *i. e.*, $3\alpha<r<\infty$, as our discussion in the last Chapters shows. Hence the radius of the circular orbits must lie between ∞ and 3α. If the radius of the orbit is equal to 3α, or six times the mass of the central body measured in the unit of length, the circular orbit reduces to a *pseudo-circular* motion (Vb), which corresponds to the case of three equal roots of the fundamental cubic. $\frac{2\omega}{\pi}$ is infinitely great in this case. As the radius grows greater, the value of $\frac{2\omega}{\pi}$ becomes smaller and the amount of the advance of the perihelion decreases. At the same time the value of the integer p in the formula $4\omega=2p\pi+\nu$ comes nearer to unity. The representative point of the motion in Fig. 13 gradually descends from C to A.

When the radius of the circle is very great compared with α, or α is very small compared with the radius, then $p=1$ and ν is very small. Thus the motion tends to the Keplerian circular motion in the limit $\frac{\alpha}{\mathbf{a}_0}\to 0$. This is the case (XV). It is very curius that the planetary orbits in our Solar System are all of the kind of trajectories named *quasi-elliptic* and indeed very near to the type (XV). This corresponds to the fact that the deviation from the Newtonain mechanics of the actual system existing in Nature is very small owing to the smallness of α compared with $\mathbf{a}_0$. If $\mathbf{a}_0$ is near to α, then the deviation from the Newtonian mechanics is remarkably great and the particle performs several revolutions between two successive perihelia or aphelia.

It is to be remarked in this connection that Whittaker's example[1] of the *singular circular orbits* can not exist physically. As stated above there is no circular orbit in the domain $r<3\alpha$. Nevertheless Whittaker treats such an orbit in the domain $r<3\alpha$ and calculates the characteristic exponents, proves that two of the four characteristic exponents do not vanish, and finally concludes the behaviour of his singular circular orbits.

4. Next we proceed to the computation of the second approximation for small values of $q=e^{-\frac{\pi\omega'}{\omega i}}$, *i. e.*, for the trajectories with small deviation from the circular.

By (69) Chap. IV and from the discussion in the case (IIb) in Chap. VI, we have

$$\left.\begin{aligned} c_0t+\beta'_3&=A_0(\psi'+\beta_1)+\sum_{n=1}^{\infty}\frac{A_nq^n}{1-q^{2n}}\sin\frac{n\pi}{2\omega}(\psi'+\beta_1),\\ u-\frac{1}{3}&=-B_0-\sum_{n=1}^{\infty}\frac{B_nq^n}{1-q^{2n}}\cos\frac{n\pi}{2\omega}\left(\psi'+\beta_1\right), \end{aligned}\right\} \quad (91)$$

where

$$\left.\begin{aligned} A_0&=\frac{\alpha}{2}\left\{\left(\zeta z-\frac{\eta z}{\omega}\right)-\frac{\wp' z}{\wp' y}\left(1-\frac{\wp'' y}{\wp'^2 y}\right)\left(\zeta y-\frac{\eta}{\omega}y\right)+\frac{\wp' z}{\wp'^2 y}\left(\frac{\eta}{\omega}-\frac{1}{3}\right)\right\},\\ A_n&=2\alpha\left\{-\frac{\sin\frac{n\pi}{\omega}z}{n}+\frac{\wp' z}{\wp' y}\left(1-\frac{\wp'' y}{\wp'^2 y}\right)\frac{\sin\frac{n\pi}{\omega}y}{n}+\frac{\wp' z}{\wp'^2 y}\frac{\pi}{\omega}\cos\frac{n\pi}{\omega}y\right\}, \end{aligned}\right. \quad (91)'$$

$$\left.\begin{aligned} B_0&=\frac{\eta}{\omega},\\ B_n&=8\left(\frac{\pi}{2\omega}\right)^2 n, \end{aligned}\right. \quad (91)''$$

$$q=e^{-\frac{\pi\omega'}{\omega i}}. \quad (91)'''$$

Inverting these infinite series (91) we get

1) E. T. Whittaker, *Analytical Dynamics*. Third Edition. (1927) p. 406.

$$\left.\begin{aligned}\psi'+\beta_1=&\frac{1}{A_0}(c_0t+\beta_3')-\frac{A_1}{A_0}\frac{q^2}{1-q^2}\sin\left(\frac{\pi}{2\omega}\cdot\frac{c_0t+\beta_3'}{A_0}\right)\\&-\left\{\frac{A_2}{A_0}\frac{q^4}{1-q^4}-\frac{1}{2}\frac{A_1^2}{A_0^2}\frac{q^4}{(1-q^2)^2}\right\}\sin\left(\frac{2\pi}{2\omega}\cdot\frac{c_0t+\beta'_3}{A_0}\right)\\&+\cdots\cdots\cdots\cdots,\end{aligned}\right\}\quad(92)$$

$$r=\frac{3\alpha}{1-3B_0}\left\{\begin{aligned}&1+\frac{3B_1}{1-3B_0}\frac{q^2}{1-q^2}\cos\left(\frac{\pi}{2\omega}\frac{c_0t+\beta'_3}{A_0}\right)\\&+\frac{1}{2}\left[\frac{3B_1}{1-3B_0}\left(\frac{q^2}{1-q^2}\right)^2\frac{A_1}{A_0}+\frac{9B_1^2}{(1-3B_0)^2}\left(\frac{q^2}{1-q^2}\right)^2\right]\\&+\left[\frac{3B_0}{1-3B_0}\cdot\frac{q^4}{1-q^4}+\frac{9B_1^2}{2(1-3B_0)^2}\cdot\frac{q^4}{(1-q^2)^4}\right.\\&\qquad\left.-\frac{6B_1}{2(1-3B_0)}\cdot\frac{A_1}{A_0}\left(\frac{q^2}{1-q^2}\right)^2\right]\cos\left(\frac{2\pi}{2\omega}\cdot\frac{c_0t+\beta'_3}{A_0}\right)\\&+\cdots\cdots\cdots\cdots.\end{aligned}\right\}\quad(93)$$

Denote the periods of ψ' with regard to t by T_0, that is,

$$T_0=\frac{2\pi A_0}{c_0}.$$

If $\frac{\pi}{2\omega}$ is very near to 1 and can be put

$$\frac{\pi}{2\omega}=1-\frac{\gamma}{2},$$

then (92) and (93) become

$$\left.\begin{aligned}\psi'+\beta_1=&\frac{1}{A_0}(c_0t+\beta_3')-\frac{A_1}{A_0}\frac{q^2}{1-q^2}\sin\left(\frac{c_0t+\beta_3'}{A_0}\right)\\&-\left\{\frac{A_2}{A_0}\frac{q^4}{1-q^4}-\frac{1}{2}\frac{A_1^2}{A_0^2}\frac{q^4}{(1-q^2)^2}\right\}\sin2\left(\frac{c_0t+\beta_3'}{A_0}\right)\\&+\cdots\cdots\cdots\cdots\\&+\frac{A_1}{A_0}\frac{q^2}{1-q^2}\frac{\gamma}{2}\cos\left(\frac{c_0t+\beta_3'}{A_0}\right)\\&-\frac{A_1}{A_0}\frac{q^2}{1-q^2}\frac{\gamma^3}{48}\cos\left(\frac{c_0t+\beta_3'}{A_0}\right)\\&+\cdots\cdots\cdots\cdots.\end{aligned}\right\}\quad(92)'$$

$$\left\{\begin{aligned}&1+\frac{3B_1}{1-3B_0}\frac{q^2}{1-q^2}\cos\left(\frac{c_0t+\beta_3'}{A_0}\right)\\&+\frac{1}{2}\left[\frac{3B_1}{1-3B_0}\left(\frac{q^2}{1-q^2}\right)^2\frac{A_1}{A_0}+\frac{9B_1^2}{(1-3B_0)^2}\left(\frac{q^2}{1-q^2}\right)^2\right]\end{aligned}\right.$$

$$r=\frac{3\alpha}{1-3B_0}\cdot\left\{\begin{array}{l} +\left[\dfrac{3B_2}{1-3B_0}\dfrac{q^4}{1-q^4}+\dfrac{9B_1^2}{2(1-3B_0)^2}\cdot\dfrac{q^4}{(1-q^2)^2}\right. \\ \qquad\left.-\dfrac{6B_1}{2(1-3B_0)}\cdot\dfrac{A_1}{A_0}\left(\dfrac{q^2}{1-q^2}\right)^2\right]\cos 2\left(\dfrac{c_0t+\beta_3'}{A_0}\right) \\ +\cdots\cdots\cdots\cdots \\ -\dfrac{3B_1}{1-3B_0}\dfrac{q^2}{1-q^2}\dfrac{\gamma}{2}\sin\left(\dfrac{c_0t+\beta_3'}{A_0}\right) \\ +\dfrac{3B_1}{1-3B_0}\dfrac{q^2}{1-q^2}\dfrac{\gamma^3}{48}\sin\left(\dfrac{c_0t+\beta_3'}{A_0}\right) \\ +\cdots\cdots\cdots\cdots \end{array}\right\} \qquad (93)'$$

Hence for very small values of γ we can regard these expressions as periodic in the period T_0 formally.

Now

$$\wp y=-\frac{1}{3}=-\frac{\eta}{\omega}+\left(\frac{\pi}{2\omega}\right)^2\cdot\frac{1}{\sin^2\dfrac{\pi y}{2\omega}}-8\left(\frac{\pi}{2\omega}\right)^2\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos\frac{n\pi y}{\omega},$$

$$\wp z=\frac{2}{3}=-\frac{\eta}{\omega}+\left(\frac{\pi}{2\omega}\right)^2\frac{1}{\sin^2\dfrac{\pi z}{2\omega}}-8\left(\frac{\pi}{2\omega}\right)^2\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos\frac{n\pi z}{\omega},$$

$$\frac{\eta}{\omega}=\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2-8\left(\frac{\pi}{2\omega}\right)^2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}.$$

By successive approximation from these series we get

$$\sin^2\chi y=-\frac{3\chi^2(1-\chi^2)}{(1-\chi^2)^2-144\chi^4q^2},$$

$$\cos^2\chi y=\frac{(1-\chi^2)(1+2\chi^2)-144\chi^4q^2}{(1-\chi^2)^2-144\chi^4q^2},$$

$$\sin^2\chi z=\frac{3\chi^2(2+\chi^2)}{(2+\chi^2)^2-144\chi^4q^2},$$

$$\cos^2\chi z=\frac{2(2+\chi^2)(1-\chi^2)-144\chi^4q^2}{(2+\chi^2)^2-144\chi^4q^2},$$

$$\sin 2\chi y=i\cdot\frac{2\sqrt{3}\,\chi\sqrt{1+2\chi^2}}{(1-\chi^2)},$$

$$\cos 2\chi y=\frac{1+5\chi^2}{1-\chi^2},$$

$$\sin 2\chi z = \frac{2\sqrt{6}\,\chi\sqrt{1-\chi^2}}{2+\chi^2},$$

$$\cos 2\chi z = \frac{2-5\chi^2}{2+\chi^2},$$

where

$$\chi = \frac{\pi}{2\omega}.$$

Substituting these values in the expansions of the elliptic functions:

$$\wp' v = -2\left(\frac{\pi}{2\omega}\right)^3 \cdot \frac{\cos\frac{\pi v}{2\omega}}{\sin^3\frac{\pi v}{2\omega}} + 16\left(\frac{\pi}{2\omega}\right)^3 \sum_{n=1}^{\infty} \frac{n^2 q^{2n}}{1-q^{2n}} \sin\frac{n\pi v}{\omega},$$

$$\wp'' v = 6\left(\frac{\pi}{2\omega}\right)^4 \frac{\cos^2\frac{\pi v}{2\omega}}{\sin^4\frac{\pi v}{2\omega}} + 2\left(\frac{\pi}{2\omega}\right)^4 \sum_{n=1}^{\infty} \frac{n^3 q^{2n}}{1-q^{2n}} \cos\frac{n\pi v}{\omega},$$

$$\zeta v = \frac{\eta v}{\omega} + \frac{\pi}{2\omega}\cot\frac{\pi v}{2\omega} + \frac{2\pi}{\omega}\sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \sin\frac{n\pi v}{\omega},$$

we obtain

$$\wp' y = -\frac{2i}{3\sqrt{3}}(1-\chi^2)\sqrt{1+2\chi^2} + i.\frac{16\sqrt{3}\,\chi^4 q^2 (5+7\chi^2)}{(1-\chi^2)\sqrt{1+2\chi^2}},$$

$$\wp'' y = \frac{2}{3}(1-\chi^2)(1+\chi^2) - 160\chi^4 q^2,$$

$$\wp' z = -\frac{2\sqrt{2}}{3\sqrt{3}}(2+\chi^2)\sqrt{1-\chi^2} + \frac{8\sqrt{6}\,\chi^4 q^2 (10-7\chi^2)}{(2+\chi^2)\sqrt{1-\chi^2}},$$

$$\zeta z - \frac{\eta z}{\omega} = \sqrt{\frac{2}{3}}\sqrt{1-\chi^2} + \sqrt{\frac{2}{3}} \cdot \frac{6(1-7\chi^2)\chi^2 q^2}{\sqrt{1-\chi^2}(2+\chi^2)},$$

$$\frac{\wp' z}{\wp' y}\left(1 - \frac{\wp'' y}{\wp'^2 y}\right) = -\frac{i}{\sqrt{2}} \cdot \frac{(2+\chi^2)(11+11\chi^2-4\chi^4)}{(1-\chi^2)^{\frac{3}{2}}(1+2\chi^2)^{\frac{3}{2}}}$$
$$- \frac{i}{\sqrt{2}} \cdot \frac{2916(1+\chi^2)(10+31\chi^2+31\chi^4)}{(1-\chi^2)^{\frac{7}{2}}(2+\chi^2)(1+2\chi^2)^{\frac{5}{2}}},$$

$$\frac{\wp' z}{\wp'^2 y} = \sqrt{\frac{2}{3}} \cdot \frac{9}{2} \cdot \frac{(2+\chi^2)}{(1-\chi^2)^{\frac{3}{2}}(1+2\chi^2)}$$
$$+ \sqrt{\frac{2}{3}} \cdot \frac{162\chi^4 q^2 (70+189\chi^2+159\chi^4+14\chi^6)}{(1-\chi^2)^{\frac{7}{2}}(1+2\chi^2)^2(2+\chi^2)},$$

$$\zeta y-\frac{\eta y}{\omega}=-\frac{i\sqrt{1+2\chi^2}}{\sqrt{3}}+i\cdot\frac{24\chi^2q^2(1+5\chi^2)}{\sqrt{3}(1-\chi^2)\sqrt{1+2\chi^2}},$$

$$\frac{\eta}{\omega}-\frac{1}{3}=-\frac{1-\chi^2}{3}-8\chi^2q^2.$$

and further by (91)

$$A_0=\frac{\alpha\sqrt{6}}{(1-\chi^2)^{\frac{3}{2}}}$$

$$\times\left[3+\frac{\chi^2q^2(1685+6510\chi^2+9369\chi^4+4772\chi^6+852\chi^8+72\chi^{10}+68\chi^{12})}{(1-\chi^2)^2(2+\chi^2)(1+2\chi^2)^2}\right],$$

$$A_1=\frac{12\sqrt{6}\,\alpha\chi(14+27\chi^2+18\chi^4)}{(1-\chi^2)^{\frac{5}{2}}(1+2\chi^2)(2+\chi^2)},$$

$$\cdots\cdots\cdots\cdots,$$

$$B_0=\frac{1}{3}\chi^2-8\chi^2q^2,$$

$$B_1=8\chi^2, \tag{94$'$}$$

$$\cdots\cdots\cdots$$

From (91) we have

$$\frac{1}{r}\cdot\frac{3\alpha}{1-\chi^2}\left(1-\frac{24\chi^2q^2}{1-\chi^2}\right)$$

$$=1-\frac{24\chi^2q}{1-\chi^2}\cos\chi(\psi'+\beta_1)$$

$$-\frac{48\chi^2q^2}{1-\chi^2}\cos2\chi(\psi'+\beta_1)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots.$$

Hence if we compare this with the usual formula for the Keplerian motion :

$$\frac{\mathbf{a}}{r}=1-\mathbf{e}\cos(\psi'+\beta_1), \tag{95}$$

we get

$$\left.\begin{aligned}\mathbf{e}&=\frac{24\chi^2q}{1-\chi^2}+\cdots\cdot,\\ \mathbf{a}&=\frac{3\alpha}{1-\chi^2}-\frac{72\alpha\chi^2q^2}{(1-\chi^2)^2}+\cdots.\end{aligned}\right\} \tag{96}$$

It is to be remarked that the expansion for $\frac{\mathbf{a}}{r}$ stops at the second term in the Newtonian mechanics, while in the relativistic mechanics the series is provided with an infinite number of terms.

Denote $\frac{2\pi A_0}{c_0}$ by T_0 as before, then

$$T_0=\frac{2\pi A_0}{c_0}=\frac{2\sqrt{6}\,\pi\alpha}{c_0\left\{1-\left(\frac{\pi}{2\omega}\right)^2\right\}^{\frac{3}{2}}}\left\{3+\frac{q^2\left(\frac{\pi}{2\omega}\right)^2\Psi\left(\frac{\pi}{2\omega}\right)}{\left[1-\left(\frac{\pi}{2\omega}\right)^2\right]^2\left[2+\left(\frac{\pi}{2\omega}\right)^2\right]\left[1+2\left(\frac{\pi}{2\omega}\right)^2\right]^2}\right\},$$

with

$$\Psi\left(\frac{\pi}{2\omega}\right)=1685+6510\left(\frac{\pi}{2\omega}\right)^2+9369\left(\frac{\pi}{2\omega}\right)^4+4772\left(\frac{\pi}{2\omega}\right)^6+852\left(\frac{\pi}{2\omega}\right)^8$$
$$+72\left(\frac{\pi}{2\omega}\right)^{10}+68\left(\frac{\pi}{2\omega}\right)^{12}. \tag{97}$$

In the limiting case in which we can write

$$\left(\frac{\pi}{2\omega}\right)^2=1-\gamma$$

with a small quantity γ, we have

$$\begin{aligned}\mathbf{e}_0&=\frac{24q}{\gamma},\\ \mathbf{a}_0&=\frac{3\alpha}{\gamma},\\ q^2&=\frac{\alpha^2\mathbf{a}_0^2}{64\mathbf{a}_0^2}=\frac{\gamma^2\mathbf{e}_0^2}{576(1-\gamma)^2},\end{aligned} \tag{98}$$

and

$$T_0=\frac{2\pi\sqrt{6}\,\mathbf{a}_0}{c_0\gamma^{\frac{1}{2}}}\left(1+\frac{1}{2}\mathbf{e}_0^2\right).$$

Hence the advance of the perihelion in one anomalistic year is

$$\pi\gamma=\frac{24\pi^3\mathbf{a}_0^2}{c_0^2T_0^2(1-\mathbf{e}_0^2)}. \tag{99}$$

This is in good agreement with Einstein's and reduces to (89) for $\mathbf{e}_0=0$.

For the general case we have, up to the order of q^2,

$$T_0=\frac{2\sqrt{2}\,\pi\mathbf{a}^{\frac{3}{2}}}{\sqrt{\alpha}\,c_0}\left\{1+\frac{\alpha^2\mathbf{e}^2}{192\,\mathbf{a}^2}\cdot\frac{\left(\frac{\pi}{2\omega}\right)^2\Psi\left(\frac{\pi}{2\omega}\right)}{\left[1-\left(\frac{\pi}{2\omega}\right)^2\right]\left[2+\left(\frac{\pi}{2\omega}\right)^2\right]\left[1+2\left(\frac{\pi}{2\omega}\right)^2\right]^2}\right\}.$$

This leads to the generalised Keplerian third law:

$$T_0^2=\frac{8\pi^2}{c_0^2}\frac{\mathbf{a}^3}{\alpha}$$

$$\times\left\{1+\frac{\alpha^2\mathbf{e}^2}{192\mathbf{a}^2}\cdot\frac{\left(\frac{\pi}{2\omega}\right)^2\Psi\left(\frac{\pi}{2\omega}\right)}{\left[1-\left(\frac{\pi}{2\omega}\right)^2\right]\left[2+\left(\frac{\pi}{2\omega}\right)^2\right]\left[1+2\left(\frac{\pi}{2\omega}\right)^2\right]^2}\right\}^2. \tag{100}$$

A more general formula for (99) is

$$T_0=\frac{2\sqrt{6}\,\pi\mathbf{a}}{c_0\left\{1-\left(\frac{\pi}{2\omega}\right)^2\right\}^{\frac{1}{2}}}\cdot$$

$$\times\left\{1+\frac{\mathbf{e}^2}{1728}\cdot\frac{\Psi\left(\frac{\pi}{2\omega}\right)}{\left(\frac{\pi}{2\omega}\right)^2\left[2+\left(\frac{\pi}{2\omega}\right)^2\right]\left[1+2\left(\frac{\pi}{2\omega}\right)^2\right]^2}\right\}. \tag{101}$$

This coincides with (90) when we make $\mathbf{e}=0$.

5. The coefficients of the first two terms of (91) are shown in (94). Hence by inverting these series we obtain

$$\left.\begin{aligned}\psi'+\beta_1&=\mathfrak{A}_0(c_0t+\beta_3')-\mathfrak{A}_1q^2\sin\left(\frac{\pi}{2\omega}\cdot\frac{c_0t+\beta_3'}{A_0}\right)+\ldots\ldots,\\ r&=\mathbf{a}\left\{1+\mathfrak{B}_1q^2\cos\left(\frac{\pi}{2\omega}\cdot\frac{c_0t+\beta_3'}{A_0}\right)+\ldots\ldots\ldots\ldots\ldots,\right.\end{aligned}\right\} \tag{102}$$

where

$$\mathfrak{A}_0=\frac{\left\{1-\left(\frac{\pi}{2\omega}\right)^2\right\}^{\frac{1}{2}}}{\sqrt{6}\,\mathbf{a}}\cdot\left\{1-\frac{\mathbf{e}^2}{1728}\cdot\frac{\Psi\left(\frac{\pi}{2\omega}\right)}{\left(\frac{\pi}{2\omega}\right)^2\left[2+\left(\frac{\pi}{2\omega}\right)^2\right]\left[1+2\left(\frac{\pi}{2\omega}\right)^2\right]^2}\right\},$$

$$\mathfrak{A}_1q^2=\frac{\mathbf{e}^2\left[1-\left(\frac{\pi}{2\omega}\right)^2\right]\left[14+27\left(\frac{\pi}{2\omega}\right)^2+18\left(\frac{\pi}{2\omega}\right)^4\right]}{144\left(\frac{\pi}{2\omega}\right)^3\left[1+2\left(\frac{\pi}{2\omega}\right)^2\right]\left[2+\left(\frac{\pi}{2\omega}\right)^2\right]^2}, \tag{102$'$}$$

$$\mathfrak{B}_1q^2=\frac{\left[1-\left(\frac{\pi}{2\omega}\right)^2\right]\mathbf{e}^2}{24\left(\frac{\pi}{2\omega}\right)^2}.$$

The arguments of the trigonometrical functions in these expressions contain $\frac{\pi}{2\omega}$ as the factors. If $\frac{\pi}{2\omega}$ is very near to 1, then our process is similar to Gylden's expansions[1] of his *absolute orbits* in his theory of absolute perturbations. Our orbits, being ellipses with moving perihelia in the approximation so far carried out, are exactly Gylden's absolute orbits, which he adopted for the intermediate orbits for the planets and for the Moon.[2] The following observation is sufficient to see the point.

$\frac{2\pi}{T_0}=\mathbf{n}$ is the mean motion in the longitude. The arguments of the trigonometrical expansions of the co-ordinates are integral multiples of

$$\frac{\pi}{2\omega}(\mathbf{n}t+\beta). \tag{103}$$

This is equivalent to

$$\mathbf{n}t-\varpi+\varepsilon,$$

with

$$\varpi=\left(1-\frac{\pi}{2\omega}\right)\mathbf{n}t,$$

$$\varepsilon=\frac{\pi}{2\omega}\beta,$$

where ϖ is the longitude of the perihelion. $1-\frac{\pi}{2\omega}$ is of the order of $\frac{\gamma}{2}$ or of $\frac{3\alpha}{\mathbf{a}}$. This quantity vanishes if we neglect the relativistic effect over the Newtonian.

6. Suppose that $\frac{2\omega}{\pi}$ is exactly equal to an integer p. Then the arguments of the trigonometrical terms in the expansions become all integral multiples of $\frac{1}{p}\left(\frac{c_0 t+\beta_3}{A_0}\right)$. Hence the motion is periodic with the period $p\ T_0$. By (101) we get

$$T_0=\frac{2\sqrt{6}\,\pi\mathbf{a}p}{c_0(p^2-1)^{\frac{1}{2}}}$$
$$\times\left\{1+\frac{\mathbf{e}^2}{1728}\cdot\frac{(1685p^{12}+6510p^{10}+9369p^8+4772p^6+852p^4+72p^2+68)}{p^4(2p^2+1)(p^2+2)}\right\}.$$

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1) H. Gylden, Acta Math. 9 (1886) 185; 15 (1891) 65; 17 (1893) 1; *Traité Analytique des Orbites Absolues des Huit Planètes Principales* (1893).

2) H. Gylden, Acta Math. 7 (1885) 125.

The period of ψ' with regard to t is T_0. Hence the particle revolves p times round the central body in one period of the motion. If we fix **a** and disregard the arbitrary constants which refer to the origin of the time and the origin of the longitude, we have an ∞^1 number of periodic solutions with a varying parameter **e**. When $\mathbf{e}=0$, the orbit reduces to be circular. The period of the circular orbit is evidently T_0. *Hence the circular orbits with the period* T_0 *in the relativistic mechanics in the case* (IIb) *are each surrounded by an infinite number of periodic orbits of the periods* $p\ T_0$. Thus this is a good example of the *periodic solutions of the second genus*[1] of Poincaré.

Chapter XI.

General Expansions for a Quasi-Elliptic Motion.

I propose to perform the complete inversion of the series:

$$\left.\begin{aligned} c_0t+\beta_3' &= A_0(\psi'+\beta_1)+\sum_{n=1}^{\infty}\frac{A_nq^n}{1-q^{2n}}\sin\frac{n\pi}{2\omega}(\psi'+\beta_1), \\ u-\frac{1}{3} &= -B_0-\sum_{n=1}^{\infty}\frac{B_nq^n}{1-q^{2n}}\cos\frac{n\pi}{2\omega}(\psi'+\beta_1), \end{aligned}\right\} \tag{91}$$

and to expand $\frac{1}{r}$ and ψ' as functions of $c_0t+\beta_3'$. It will be proved that such expansions are trigonometrical series with integral multiples of $\frac{\pi}{2\omega}\frac{c_0t+\beta_3'}{A_0}$ as the arguments.

1. Consider an expression of the form:

$$t=kv-\alpha_1\sin v-\alpha_2\sin 2v-\dots\dots-\alpha_m\sin mv. \tag{104}$$

Let the inverse function of this expansion be in the form:

$$v=\frac{t}{k}+\sum_{j=1}^{\infty}C_j\sin\frac{jt}{k}. \tag{105}$$

In order that this inversion be possible, the differential coefficient:

$$\frac{\partial t}{\partial v}=k-\alpha_1\cos v-2\alpha_2\cos 2v-\dots\dots-m\alpha_m\cos mv, \tag{106}$$

ought not to vanish for any value of v in the domain under our con-

1) H. P. Poincaré, *Méthodes Nouvelles de la Mécanique Céleste*. T. 3. (1899); Hagihara, Japanese Journal of Astronomy and Geophysics. 5 (1927) 1.

sideration, by a theorem in the theory of implicit functions. But it is sufficient for the non-vanishing of $\frac{\partial t}{\partial v}$ that we should have a relation:

$$k > |\alpha_1| + 2|\alpha_2| + \cdots\cdots + m|\alpha_m|. \tag{107}$$

We assume that this inequality holds throughout our discussion.

By differentiating (105) we get

$$\sum_{j=1}^{\infty} \frac{j}{k} C_j \cos\frac{jt}{k} = \frac{dv}{dt} - \frac{1}{k}.$$

If we multiply this expression by $\cos\frac{jt}{k}\,dt$ and integrate the result over the interval $0 \leqq t \leqq \pi$, then

$$\frac{1}{2}\pi j C_j = \int_0^\pi \left(\frac{dv}{dt} - \frac{1}{k}\right)\cos\frac{jt}{k}\,dt = \int_0^\pi \frac{dv}{dt}\cos\frac{jt}{k}\,dt.$$

As $v=0$ for $t=0$ and $v=\pi$ for $t=\pi$, we can change the independent variable from t to v.

$$\frac{1}{2}\pi j\, C_j = \int_0^\pi \cos\frac{jt}{k}\,dv = \int_0^\pi \cos j\left(v - \sum_{l=1}^{m}\frac{\alpha_l}{k}\sin lv\right)dv.$$

Hence

$$C_j = \frac{2}{j\pi}\int_0^\pi \cos\left(jv - j\sum_{l=1}^{m}\frac{\alpha_l}{k}\sin lv\right)dv. \tag{108}$$

Define the Bessel functions of m variables:[1)]

$$e^{\sum_{j=1}^{m}\frac{x_j}{2}\left(v^j - v^{-j}\right)} = \sum_{l=-\infty}^{\infty} (x_1, x_2, \ldots\ldots, x_m) v^l. \tag{109}$$

Write $\frac{1}{v}$ instead of v and $-x_j$ instead of x_j, then

$$e^{\sum_{j=1}^{m}\frac{x_j}{2}\left(v^j - v^{-j}\right)} = \sum_{l=-\infty}^{\infty} J_l(-x_1, -x_2, \ldots\ldots, -x_m) v^{-l},$$
$$= \sum_{l=-\infty}^{\infty} J_{-l}(x_1, x_2, \ldots\ldots, x_m)\, v^l,$$

1) P. Appell, C.R. **160** (1915) 419; **179** (1924) 437; Sur les Fonctions Hypergéométriques de Plusieures Variables, les Polynomes d'Hermite et Autres Fonctions Sphériques dans l'Hyperespace. (1925); J. Pérès, C.R. **161** (1915) 168; M. Akimoff, C.R. **163** (1916) 26; **165** (1917) 23; **179** (1924) 435; **185** (1927) 409, 435; B. Jekhowsky, C.R. **162** (1916) 318; Bull. Soc. Math. France. [ii] **41** (1917) 58; C.R. **164** (1917) 719; **170** (1920) 1042; Bull. Astr. **35** (1918) 139, 145; C.R. **172** (1921) 1331; Journal des Observateurs. **10** (1927) No. 7 et No. 9; Humbert, Proc. Roy. Soc. Edinburgh. **41** (1920) 73.

$$=\sum_{l=-\infty}^{\infty} J_l\,(x_1, x_2, \ldots, x_m)v^l.$$

Hence we get a relation:

$$J_{-l}(-x_1, -x_2 \ldots, -x_m)=J_l\,(x_1, x_2, \ldots x_m). \tag{110}$$

Next write $\frac{1}{v}$ instead of v and particularly put $v=e^{\varphi\sqrt{-1}}$, and separate the real and the imaginary parts.

$$\left.\begin{aligned}
&\cos\Big(\sum_{j=1}^{m} x_j \sin j\varphi\Big)\\
&\qquad=\sum_{l=1}^{\infty}[J_l(x_1, x_2, \ldots, x_m)+J_{-l}(x_1, x_2, \ldots, x_m)]\cos l\,\varphi,\\
&\sin\Big(\sum_{j=1}^{m} x_j \sin j\varphi\Big)\\
&\qquad=\sum_{l=1}^{\infty}[J_l(x_1, x_2, \ldots, x_m)-J_{-l}\,(x_1, x_2, \ldots, x_m)]\sin l\varphi.
\end{aligned}\right\} \tag{111}$$

Multiply these two expressions by $\begin{matrix}\cos\\ \sin\end{matrix}\; n\varphi$ respectively and integrate the result between o and π, then we have (n = integer)

$$\left.\begin{aligned}
&\int_0^{\pi} \cos n\varphi \cos\Big(\sum_{j=1}^{m} x_j \sin j\varphi\Big)d\varphi\\
&\qquad=\frac{\pi}{2}[J_n(x_1, x_2, \ldots, x_m)+J_{-n}\,(x_1, x_2, \ldots, x_m)],\\
&\int_0^{\pi} \sin n\varphi \sin\Big(\sum_{j=1}^{m} x_j \sin j\varphi\Big)d\varphi\\
&\qquad=\frac{\pi}{2}[J_n(x_1, x_2, \ldots, x_m)-J_{-n}(x_1, x_2, \ldots, x_m)].
\end{aligned}\right\} \tag{112}$$

Hence we get

$$\int_0^{\pi} \cos\Big(n\varphi-\sum_{j=1}^{m} x_j \sin j\varphi\Big)\,d\varphi=\pi J_n\,(x_1, x_2, \ldots, x_m). \tag{113}$$

If we apply these formulae to (108), we have

$$C_j=\frac{2}{j}J_j\Big(\frac{j\alpha_1}{k}, \frac{j\alpha_2}{k}, \ldots, \frac{j\alpha_m}{k}\Big).$$

Hence

$$v=\frac{t}{k}+\sum_{j=1}^{\infty}\frac{2}{j}J_j\Big(\frac{j\alpha_1}{k}, \frac{j\alpha_2}{k}, \ldots, \frac{j\alpha_m}{k}\Big)\sin\frac{jt}{k}\,. \tag{114}$$

2. Put

$$t_m = kv_m - \sum_{j=1}^{m} \alpha_j \sin jv_m + \mu_m,$$

and

$$v_m = \frac{t_m}{k} + \sum_{j=1}^{\infty} \frac{2}{j} J_j\left(\frac{j\alpha_1}{k}, \frac{j\alpha_2}{k}, \ldots, \frac{j\alpha_m}{k}\right) \sin \frac{jt_m}{k} + \delta v_m.$$

Then

$$\delta v_m = \frac{\mu_m}{k} + \sum_{j=1}^{\infty} \frac{2}{j}. \ J_j\left(\frac{j\alpha_1}{k}, \ldots, \frac{j\alpha_m}{k}\right) \left[\sin \frac{(jt_m + j\mu_m)}{k} - \sin \frac{jt_m}{k}\right]$$

$$= \frac{\mu_m}{k} + 4\sum_{j=1}^{\infty} \frac{1}{j}. \ J_j\left(\frac{j\alpha_1}{k}, \ldots, \frac{j\alpha_m}{k}\right) \sin \frac{j\mu_m}{k} \cos\left(\frac{jt_m}{k} + \frac{j\mu_m}{k}\right)$$

$$< \text{Max.} \left|\frac{\partial v_m}{\partial t_m}\right|. \ \mu_m.$$

As $\left|\frac{\partial v_m}{\partial t_m}\right|$ is finite for all values of t_m by (107), we can make δv_m as small as we please by choosing μ_m sufficiently small. Suppose that (107) holds for all integral values of m, that is, for $m = 1, 2, \ldots$, and that $|\alpha_m|$ decreases towards zero as m increases indefinitely. Then, by taking m sufficiently great, the inversion of

$$t = kv - \alpha_1 \sin v - \alpha_2 \sin 2v \ldots - \alpha_m \sin mv - \ldots,$$

can be approximated as nearly as we please by a function:

$$v = \frac{t}{k} + \sum_{j=1}^{\infty} \frac{2}{j}. \ J_j\left(\frac{j\alpha_1}{k}, \frac{j\alpha_2}{k}, \ldots, \frac{j\alpha_m}{k}\right) \sin \frac{jt}{k}.$$

Hence $v - \frac{t}{\text{k}}$ is expanded in purely trigonometrical series with multiples of $\frac{t}{k}$ as the arguments, provided that we disregard the convergency of the expansion. This expansion, however, has a semi-convergent character, as δv_m vanishes with μ_m and μ_m tends to zero when we make $m \to \infty$.

3. Put

$$\cos pv = \frac{1}{2} D_0 + \sum_{j=1}^{\infty} D_j \cos \frac{jt}{k},$$

then

$$D_j = \frac{2}{\pi} \int_0^{\pi} \cos pv \cos \frac{jt}{k} \, dt.$$

Integrating this by parts, we have

$$D_j=\frac{2pk}{\pi j}\int_0^\pi \cos pv \sin\frac{jt}{k}\,dv$$

$$=\frac{2pk}{\pi j}\int_0^\pi \sin pv \sin j\left(v-\sum_{l=1}^{m}\frac{\alpha_l}{k}\sin lv\right)dv$$

$$=\frac{pk}{\pi j}\int_0^\pi \cos\left(jv-pv-j\sum_{l=1}^{m}\frac{\alpha_l}{k}\sin lv\right)dv$$

$$-\frac{pk}{\pi j}\int_0^\pi \cos\left(jv+pv-j\sum_{l=1}^{m}\frac{\alpha_l}{k}\sin lv\right)dv.$$

Therefore

$$D_j=\frac{pk}{j}\left[J_{j-p}\left(\frac{j\alpha_1}{k},\frac{j\alpha_2}{k},\ldots,\frac{j\alpha_m}{k}\right)-J_{j+p}\left(\frac{j\alpha_1}{k},\frac{j\alpha_2}{k},\ldots,\frac{j\alpha_m}{k}\right)\right].$$

Especially if $j=0$, we have

$$D_0=\frac{2}{\pi}\int_0^\pi \cos pv\left(k-\sum_{l=1}^{m} l\alpha_l \cos lv\right)dv=-p\alpha_p.$$

Hence

$$\left.\begin{aligned}
\cos pv&=-\frac{p\alpha_p}{2}\\
&+pk.\sum_{j=1}^{\infty}\left[J_{j-p}\left(\frac{j\alpha_1}{k},\ldots,\frac{j\alpha_m}{k}\right)-J_{j+p}\left(\frac{j\alpha_1}{k},\ldots,\frac{j\alpha_m}{k}\right)\right]\frac{\cos\frac{jt}{k}}{j}.\\
&\text{Similarly}\\
\sin pv&=pk\sum_{j=1}^{\infty}\left[J_{j-p}\left(\frac{j\alpha_1}{k},\ldots,\frac{j\alpha_m}{k}\right)-J_{j+p}\left(\frac{j\alpha_1}{k},\ldots,\frac{j\alpha_m}{k}\right)\right]\frac{\sin\frac{jt}{k}}{j}.
\end{aligned}\right\}\quad(115)$$

4. Now let us apply these formulae to our problem (91). We have

$$t+\frac{\beta_3'}{c_0}=\left(\frac{2\omega}{\pi}\frac{A_0}{c_0}\right)\frac{\pi}{2\omega}(\psi'+\beta_1')+\sum_{n=1}^{m}\frac{A_n}{c_0}\frac{q^n}{1-q^{2n}}\sin\frac{n\pi}{2\omega}(\psi'+\beta_1).$$

Put
$$\frac{\pi}{2\omega}(\psi'+\beta_1)=v,\qquad \frac{2\omega}{\pi}\frac{A_0}{c_0}=k,$$

$$\frac{A_n}{c_0}\frac{q^n}{1-q^{2n}}=-\alpha_n,\quad t+\frac{\beta'_3}{c_0}=t',$$

then we get

$$t'=kv-\sum_{n=1}^{m}\alpha_n \sin nv.$$

Inverting this by the above method (114), we obtain

$$\psi'+\beta_1=\frac{c_0t+\beta'_3}{A_0}$$

$$-\frac{4\omega}{\pi}\cdot\sum_{j=1}^{m}\frac{1}{j}J_j\left(\frac{\pi A_1q}{2\omega A_0(1-q^2)},\frac{\pi A_2q^2}{2\omega A_0(1-q^4)},\ldots,\frac{\pi A_mq^m}{2\omega A_0(1-q^{2m})}\right)$$

$$\times\sin\left(\frac{j\pi(c_0t+\beta'_3)}{2\omega A_0}\right). \qquad (116)$$

Thus ψ' is expressed in terms of t.

Next by (115)

$$\cos p\frac{\pi}{2\omega}(\psi'+\beta)=\frac{pA_pq^p}{2c_0(1-q^{2p})}$$

$$+p\frac{2\omega A_0}{\pi c_0}\cdot\sum_{j=1}^{\infty}\left[J_{j-p}\left(\frac{\pi A_1q}{2\omega A_0(1-q^2)},\ \ldots\ ,\frac{\pi A_mq^m}{2\omega A_0(1-q^{2m})}\right)\right.$$

$$\left.-J_{j+p}\left(\frac{\pi A_1q}{2\omega A_0(1-q^2)},\ldots,\frac{\pi A_mq^m}{2\omega A_0(1-q^{2m})}\right)\right]\frac{\cos\frac{j\pi(c_0t+\beta_3')}{2\omega A_0}}{j},$$

$$\sin p\frac{\pi}{2\omega}(\psi'+\beta_1)=-p\frac{2\omega A_0}{\pi c_0}\cdot\sum_{j=1}^{\infty}\left[J_{j-p}\left(\frac{\pi A_1q}{2\omega A_0(1-q^2)},\ \ldots\ ,\frac{\pi A_mq^m}{2\omega A_0(1-q^{2m})}\right)\right.$$

$$\left.-J_{j+p}\left(\frac{\pi A_1q}{2A\omega_0(1-q^2)},\ \ldots,\frac{\pi A_mq^m}{2\omega A_0(1-q^{2m})}\right)\right]\frac{\sin\frac{j\pi(c_0t+\beta'_3)}{2\omega A_0}}{j}.$$

Hence

$$u-\frac{1}{3}+B_0=-\sum_{p=1}^{\infty}\frac{B_pq^p}{1-q^{2p}}\cos p\left(\frac{\pi}{2\omega}(\psi'+\beta_1)\right),$$

or

$$u=\mathfrak{B}_0+\sum_{j=1}^{\infty}\mathfrak{B}_j\cos j\left(\frac{\pi}{2\omega}(\psi'+\beta_1)\right),$$

where

$$\mathfrak{B}_0=\sum_{p=1}^{\infty}\frac{pA_pB_p}{2}\left(\frac{q^p}{1-q^{2p}}\right)^2+\frac{1}{3}-B_0,$$

$$\mathfrak{B}_j=-\frac{1}{j}\sum_{p=1}^{\infty}\frac{B_pq^p}{1-q^{2p}}\left[J_{j-p}\left(\frac{\pi A_1q}{2\omega A_0(1-q^2)},\ldots,\frac{\pi A_mq^m}{2\omega A_0(1-q^{2m})}\right)\right.$$

$$\left.-J_{j+p}\left(\frac{\pi A_1q}{2\omega A_0(1-q^2)},\ldots,\frac{\pi A_mq^m}{2\omega A_0(1-q^{2m})}\right)\right].$$

Hence $\mathbf{a}=\frac{\alpha}{\mathfrak{B}_0}$ is Gylden's *protometer* and $\mathbf{e}=\frac{\mathfrak{B}_1}{\mathfrak{B}_0}$ is his *diastematic modulus* in the approximation so far obtained.

Thus $\frac{1}{r}$ and $(\psi'+\beta_1)-\frac{c_0t+\beta_3'}{A_0}$ are periodic with the period $\frac{4\omega A_0}{c_0}$ in t and can be expressed in purely trigonometrical series with multiples of $\frac{\pi(c_0t+\beta_3')}{2\omega A_0}$ as the arguments.

5. To those who are not contented themselves with the above semi-convergent process, the following method of procedure is recommended.

We are tempted to generalise the Bessel functions $J_j(x_1, x_2, \ldots, x_m)$ to the case of an infinite number of parameters $x_1, x_2, x_3, \ldots, x_m, \ldots$. If $\sum_{j=1}^{\infty} x_j(v^j-v^{-j})$ is absolutely and uniformly convergent, then $e^{\sum_{j=1}^{\infty}\frac{x_j}{2}(v^j-v^{-j})}$ represents a definite holomorphic function in a ring domain and it can evidently be expanded into a convergent Laurent series:

$$\sum_{j=-\infty}^{\infty} J_j(x_1, x_2, \ldots, x_m, \ldots)v^j,$$

in the same ring domain. $J_j(x_1, x_2, \ldots, x_m, \ldots)$ is merely a functional symbol representing the coefficient of the expansion. Thus $J_j(x_1, x_2, \ldots, x_m, \ldots)$ is defined by

$$e^{\sum_{j=1}^{\infty}\frac{x_j}{2}(v^j-v^{-j})}=\sum_{l=-\infty}^{\infty} J_l(x_1, x_2, \ldots, x_m, \ldots)v^l,$$

or by

$$\cos\left(\sum_{j=1}^{\infty} x_j \sin j\varphi\right)=\sum_{l=1}^{\infty}[J(x_1, x_2, \ldots, x_m, \ldots)+J_{-l}(x_1, \ldots, x_m, \ldots)] \cos l\varphi,$$

$$\sin\left(\sum_{j=1}^{\infty} x_j \sin j\varphi\right)=\sum_{l=1}^{\infty}[J_l(x_1, \ldots, x_m, \ldots)-J_{-l}(x_1, \ldots, x_m, \ldots)] \sin l\varphi.$$

As we have assumed that the series $\sum_{j=1}^{\infty} |x_j|$ is absolutely convergent, the series $\sum_{j=1}^{\infty} x_j \sin j\varphi$ represents a function of φ, periodic in φ and never becomes infinite in the whole domain of φ. Hence $\frac{\cos}{\sin}\left(\sum_{j=1}^{\infty} x_j \sin j\varphi\right)$ represents also a function of φ periodic in φ and never becomes infinite in the whole domain of φ. Hence the function can be expanded into Fourier series of the above form by the well-known classical theorem of Fourier.

The expansibility is the more certain, if we remark the following inequalities.

By multiplying two expressions:

$$e^{\sum_{j=1}^{\infty}\frac{x_j}{2}(v^j-v^{-j})}=\sum_{l=-\infty}^{\infty} J_l(x_1, x_2, \ldots, x_m, \ldots)v^l,$$

$$e^{-\sum_{j=1}^{\infty}\frac{x_j}{2}(v^j-v^{-j})}=\sum_{l=-\infty}^{\infty} J_l(x_1, x_2, \ldots, x_m, \ldots)v^{-l},$$

side by side, we get a relation of the form:

$$1=E_0+\sum_{l=1}^{\infty} E_l v^l+\sum_{l=1}^{\infty} E_{-l}v^{-l}.$$

This holds for all values of v in the interval $0<|v|<1$. Hence we ought to have

$$E_0=1, \quad E_l=0, \quad E_{-l}=0.$$

The first relation, when expanded, gives

$$1=\sum_{l=-\infty}^{\infty} J_l^2(x_1, x_2, \ldots, x_m, \ldots).$$

From this we get

$$|J_l(x_1, x_2, \ldots, x_m, \ldots)|<1,$$

and $$|J_l^2(x_1, x_2, \ldots, x_m, \ldots)+J_{-l}^2(x_1, x_2, \ldots, x_m, \ldots)|<1,$$

for all integral values of l.

The difficulty lies only in the computation of the function $J_l(x_1, x_2, \ldots, x_m, \ldots)$.

To compute this function from the ordinary Bessel functions $J_l(x)$, we start with Jekhowsky's formula:

$$\begin{aligned} J_l(x_1, x_2, \ldots, x_m) &= J_0(x_m)J_l(x_1, \ldots, x_{m-1}) \\ &+\sum_{p=1}^{\infty} J_{2p}(x_m)[J_{l-2pm}(x_1, \ldots, x_{m-1})+J_{l+2pm}(x_1, \ldots, x_{m-1})] \\ &+\sum_{p=1}^{\infty} J_{2p-1}(x_m)[J_{l-(2p-1)m}(x_1, \ldots, x_m)-J_{l+(2p-1)m}(x_1, \ldots, x_m)], \end{aligned}$$

in which[1)]

$$J_0(x)=1-\frac{x}{2^2}+\frac{x^4}{2^2.4^2}-\ldots,$$

1) Watson, *A Treatise on the Theory of Bessel Functions.* (1922)

$$J_n(x)=\frac{x^n}{2^n.n!.}\left\{1-\frac{x^2}{2^2.1.(n-1)}+\frac{x^4}{2^4.1.2.(n+1)(n+2)}-\cdots\right\}.$$

Hence in the limit $x_m\to 0$ we have

$$J_l(x_1,\ \ldots,x_m)\to J_l(x_1,\ \ldots,x_{m-1}).$$

This provides a new proof of the proposition in § 3.

Jekhowsky obtained a relation:

$$J_l(x_1,\ \ldots,x_m)=\sum_{q_m=-\infty}^{\infty}J_{l-mq_m}(x_1,\ \ldots,x_{m-1})J_{q_m}(x_m)$$

$$=\sum_{q_2=-\infty}^{\infty}\sum_{q_3=-\infty}^{\infty}\cdots\sum_{q_m=-\infty}^{\infty}J_{l-2q_2-3q_3\cdots-mq_m}(x_1)J_{q_2}(x_2)\cdots J_{q_m}(x_m).$$

The computation of $J_l(x_1,\ \ldots,x_m,\ \ldots)$ by this method involves an infinite process and the detailed discussion is postponed to future.

Similar formulae for $\frac{\cos}{\sin}pv$ to (115) are obtained for this case and the results for r and for ψ' are only modified by putting $x_1,x_2,\ \ldots,x_m,\ \ldots$ instead of $x_1,x_2,\ \ldots,x_m$ in the arguments of $J_l(x_1,x_2,\ \ldots,x_m)$.

It is interesting to remark in this connection that Jekhowsky's generalisation of Cauchy's theorem, which is of use in the expansions for a quasi-elliptic motion, can be stated, in our case of an infinite number of parameters, as follows:

Consider a finite determinate function S expanded in the form

$$S=\sum_{l=-\infty}^{\infty}P_l z^l.$$

The coefficient P_l is equal to the coefficient of s^l in the expansion of a function

$$U=Se^{\frac{l}{2k}\sum_{j=1}^{\infty}\alpha_j(s^j-s^{-j})}\left[1-\frac{1}{2}\sum_{j=1}^{\infty}j\alpha_j'(s^j+s^{-j})\right],$$

in powers of s and also it is equal to the coefficient of s^{l-1} in the expansion of

$$V=\frac{k}{l}\frac{dS}{ds}e^{\frac{l}{2k}\sum_{j=1}^{\infty}\alpha_j(s^j-s^{-j})}$$

in powers of s.

SUMMARY.

1. The variational principle

$$\delta\int ds=0, \qquad ds^2=\sum_{\alpha,\beta}^{1,2,3,4} g_{\alpha\beta}\, dx_\alpha\, dx_\beta,$$

is proved to be equivalent to

$$\delta\int Td\sigma=0, \qquad T=\frac{1}{2}\sum_{\alpha,\beta}^{1,2,3,4} g_{\alpha\beta}\frac{dx_\alpha}{d\sigma}\frac{dx_\beta}{d\sigma},$$

with an additional condition that, after determining x_i as functions of σ by treating the latter variational principle as though there were no restriction at all, we should determine a constant C^2 by the relation:

$$\sum_{\alpha,\beta}^{1,2,3,4} g_{\alpha\beta}\frac{dx_\alpha}{d\sigma}\frac{dx_\beta}{d\sigma}=C^2,$$

as a function of the constants of integration, and finally put $s=C\sigma$ with this value of C. Here we can choose $C^2=1$ at the outset. If we choose $C^2=0$, then the trajectories thus obtained are those for the light rays.

2. If we form the characteristics of the Hamilton-Jacobi partial differential equation of our problem, then those characteristics can be transformed to the equations for the relativistic trajectories of a particle in the classical treatment of the problem. The singularity in the variational problem does not occur in the actual trajectories.

3. The Hamilton-Jacobi partial differential equation can be integrated by the method of the separation of the variables. They can be reduced to

$$\left.\begin{aligned} \frac{h_1}{\alpha^2}ds&=\frac{du}{u^2\sqrt{U(u)}}, \\ 0&=\frac{du}{\sqrt{U(u)}}-d\psi, \\ 0&=\frac{du}{(1-u)u^2\sqrt{U(u)}}-\frac{h_1c_0}{\alpha^2h_3}dt, \end{aligned}\right\}$$

with

$$U(u)=u^3-u^2+\frac{\alpha^2}{h_1^2}u+\frac{\alpha^2(h_3^2-1)}{h_1^2},$$

where c_0 and α are the constants in Schwarzschild's line element, h_1 and h_3 are the constants of integration, s, t, ψ and u are the proper time,

the universal time, the argument of latitude, and the constant α divided by the radius vector, respectively. The integration of these equations provides us with two constants of integration, β_1 and β_3, conjugate to h_1 and h_3. The other pair, h_2 and β_2, enters only to specify the position of the orbital plane. We do not count the constants corresponding to the proper time.

4. By choosing y and z so that

$$\wp y = -\frac{1}{3}, \qquad \wp z = \frac{2}{3},$$

the result of integration is

$$\frac{h_1}{2\alpha^2}(s+\beta_0) = \frac{1}{\wp'^2 y}\Bigg[-\zeta\left(\frac{\psi+\beta_1}{2}+y\right)-\zeta\left(\frac{\psi+\beta_1}{2}-y\right)-(\psi+\beta_1)\wp y$$

$$+\frac{\wp'' y}{\wp' y}\left\{\log\frac{\sigma\left(\frac{\psi+\beta_1}{2}+y\right)}{\sigma\left(\frac{\psi+\beta_1}{2}-y\right)}-(\psi+\beta_1)\zeta y\right\}\Bigg],$$

$$\frac{h_1}{\alpha^2 h_3}(c_0 t+\beta_3) = \frac{1}{\wp' z}\left\{(\psi+\beta_1)\zeta z - \log\frac{\sigma\left(\frac{\psi+\beta_1}{2}+z\right)}{\sigma\left(\frac{\psi+\beta_1}{2}-z\right)}\right\}$$

$$-\frac{1}{\wp' y}\left\{(\psi+\beta_1)\zeta y - \log\frac{\sigma\left(\frac{\psi+\beta_1}{2}+y\right)}{\sigma\left(\frac{\psi+\beta_1}{2}-y\right)}\right\}\left(1-\frac{\wp'' y}{\wp'^2 y}\right)$$

$$+\frac{1}{\wp'^2 y}\left\{\zeta\left(\frac{\psi+\beta_1}{2}+y\right)+\zeta\left(\frac{\psi+\beta_1}{2}-y\right)+(\psi+\beta_1)\wp y\right\},$$

$$u-\frac{1}{3} = \wp\left(\frac{\psi+\beta_1}{2}\right).$$

The radius vector r, the longitude θ and the latitude φ can be obtained by the formulae:

$$r = \frac{\alpha}{u},$$

$$\tan(\theta+\beta_2) = \tan\psi . \cos I,$$

$$\cos\varphi = \sin I . \sin\psi,$$

where

$$\cos I = \frac{h_2}{h_1}.$$

5. Four discrete cases of the distribution of the roots of the fundamental cubic $U(u)=0$ among the singularities $u=0$ and $u=1$ are distinguished. By writing $\lambda=\frac{\alpha^2}{h_1^2}$, $\mu=h_3^2$, the domains of the various cases are shown in Fig. 2.

6. The following types of motion are obtained in the non-degenerate cases of the elliptic functions. The roots of the fundamental cubic are $e_3+\frac{1}{3}<e_2+\frac{1}{3}<e_1+\frac{1}{3}$, when they are real. (See, Figs. 3–8)

Quasi-elliptic $\qquad \frac{\alpha}{e_2+\frac{1}{3}}\leqq r\leqq \frac{\alpha}{e_3+\frac{1}{3}}$,

Quasi-parabolic $\qquad \frac{\alpha}{e_2+\frac{1}{3}}\leqq r\leqq \infty$,

Quasi-hyperbolic $\qquad \frac{\alpha}{e_2+\frac{1}{3}}\leqq r<\infty$,

Pseudo-elliptic $\qquad \alpha<r\leqq \frac{\alpha}{e_1+\frac{1}{3}}$,

Pseudo-parabolic $\qquad \alpha<r\leqq\infty$,

Pseudo-hyperbolic $\qquad \alpha<r<\infty$.

The limits of the domains with only inequality signs in this table are those of asymptotic approaches. This asymptotic character is such that the moving particle tends to a standing still at a definite point on the circle $r=\alpha$ as the time tends to $+\infty$ and at another definite point on the same circle as the time tends to $-\infty$. The motion inside the circle $r=\alpha$ is inadmissible from the principle of relativity. Even the circle $r=\alpha$ is always situated inside a star of physically possible density and the orbits tending to $r=\alpha$ from outside are physically collisional or ejectional.

Each of the two trajectories, the quasi-hyperbolic and the pseudo-hyperbolic, approaches to two distinct asymptotes, one for $t\rightarrow+\infty$ and the other for $t\rightarrow-\infty$. For the quasi-parabolic and the pseudo-parabolic types these two asymptotes coincide on one straight line.

7. For the degenerate cases of the elliptic functions we get some very interesting examples in dynamics. (See, Figs. 9–12)

Quasi-elliptic spiral $\dfrac{\alpha}{e_A+\dfrac{1}{3}}<r\leqq\dfrac{\alpha}{e_3+\dfrac{1}{3}}$, $e_A=e_1=e_2$,

Quasi-parabolic spiral $\dfrac{\alpha}{e_A+\dfrac{1}{3}}<r\leqq\infty$,

Quasi-hyperbolic spiral $\dfrac{\alpha}{e_A+\dfrac{1}{3}}<r<\infty$,

Pseudo-spiral $\alpha<r<\dfrac{\alpha}{e_B+\dfrac{1}{3}}$, $e_B=e_2=e_3$,

Pseudo-circular $r=3\alpha$, $e_1=e_2=e_3=0$,

Circular $r=\dfrac{\alpha}{e_B+\dfrac{1}{3}}$.

The asymptotic approach to $\dfrac{\alpha}{e_A+\dfrac{1}{3}}$ is such that the particle performs an infinite number of revolutions, tending to the circle indefinitely as $t\rightarrow+\infty$ or as $t\rightarrow-\infty$. The asymptotic approach to $r=\alpha$ is the same as before. The pseudo-circular type is the limiting type of the pseudo-spiral and it can exist. This type of motion occurs only when the three roots of the fundamental cubic coincide at $r=3\alpha$. The approach to the limiting circle is algebraic, while in the other cases it is transcendental. The limiting circular motion other than the pseudo-circular does not exist. The circular is the only type which is in connection with the Newtonian trajectories. The quasi-elliptic spiral is *doubly asymptotic* in the sense of Poincaré. The circle $r=\dfrac{\alpha}{e_A+\dfrac{1}{3}}$ constitutes a *cycle limte* of Poincaré. Such a motion can occur for the values of r between $2\alpha<r<3\alpha$. As the circle $r=\alpha$, or even the circle $r=3\alpha$, is situated deep in the interior of a star of physically possible density, these interesting features unfortunately can not be physically realisable, if Stoner's limit for a star's density is accepted.

8. The real expansions with real arguments for the co-ordinates are also given for all these various types of motion.

9. The types of motion corresponding to different points in Fig. 2 or in Fig. 13 are completely dealt with. The transitions of the types of motion as we vary the parameters λ and μ are treated in detail. (See, Figs. 13–14.)

10. The trajectory of a light ray is found to be the type, quasi-hyperbolic, or pseudo-hyperbolic, or quasi-hyperbolic spiral, according to the circumstances. For a physically existing star the pseudo-elliptic type does not exist, though it can be imagined. Hence the statement that a very massive star can entirely absorb the light emitted from its surface and never be seen from outside, is quite fallacious. Even the third type, the quasi-hyperbolic spiral, can not exist. Thus any observer can see every physically existing star in the Universe.

11. A quasi-elliptic trajectory, endowed with the property of quasi-periodic functions, covers the whole area of its domain of motion *every-where densely*. There are two disposable constants, $\frac{\pi}{2\omega}$ and $q=e^{-\frac{\pi\omega'}{\omega i}}$, corresponding to the period of the motion and to the amplitude between the maximum and the minimum radii vectors, respectively. Supposing that $\frac{\omega'}{i}$ is fairly great and $\frac{\pi}{2\omega}$ is nearly unity, we have Einstein's formula for the advance of the perihelion:

$$\frac{24\pi^3\mathbf{a}_0^2}{c_0^2T_0^2(1-\mathbf{e}_0^2)}.$$

For a general value of $\frac{\pi}{2\omega}$ but with a fairly large values of $\frac{\omega'}{i}$, we have

$$T_0=\frac{2\sqrt{6}\,\pi\mathbf{a}}{c_0\left\{1-\left(\frac{\pi}{2\omega}\right)^2\right\}^{\frac{1}{2}}}\left\{1+\frac{\mathbf{e}^2}{1728}\cdot\frac{\Psi\left(\frac{\pi}{2\omega}\right)}{\left(\frac{\pi}{2\omega}\right)^2\left[2+\left(\frac{\pi}{2\omega}\right)^2\right]\left[1+2\left(\frac{\pi}{2\omega}\right)^2\right]^2}\right\},$$

where

$$\Psi\left(\frac{\pi}{2\omega}\right)=1685+6510\left(\frac{\pi}{2\omega}\right)^2+9369\left(\frac{\pi}{2\omega}\right)^4+4772\left(\frac{\pi}{2\omega}\right)^6$$
$$+852\left(\frac{\pi}{2\omega}\right)^8+72\left(\frac{\pi}{2\omega}\right)^{10}+68\left(\frac{\pi}{2\omega}\right)^{12}.$$

The generalised Kepler's third law is

$$T_0'^2=\frac{8\pi^2}{c_0^2}\frac{\mathbf{a}^3}{\alpha}\left\{1+\frac{\alpha^2\mathbf{e}^2}{192\mathbf{a}^2}\cdot\frac{\left(\frac{\pi}{2\omega}\right)^2\Psi\left(\frac{\pi}{2\omega}\right)}{\left[1-\left(\frac{\pi}{2\omega}\right)^2\right]\left[2+\left(\frac{\pi}{2\omega}\right)^2\right]\left[1+2\left(\frac{\pi}{2\omega}\right)^2\right]^2}\right\}^2.$$

a and **e** correspond respectively to the semi-major axis and the eccentricity in the Keplerian orbit. The orbits thus obtained are identified with Gylden's *absolute orbits*.

When $\frac{2\omega}{\pi}$ is exactly equal to an integer, then the corresponding circular orbit is proved to be a *periodic solution of the second genus* of Poincaré.

Whittaker's example of the unstable *singular circular orbits* is proved not to exist.

12. In order to write down the complete expressions of the co-ordinates for a quasi-elliptic motion in trigonometrical series of t, *Bessel functions with several arguments* are introduced. $\psi' + \beta_1 - \frac{c_0 t + \beta_3}{A_0}$ and $\frac{1}{r}$ are expressed in purely trigonometrical series with integral multiples of $\frac{\pi(c_0 t + \beta_3)}{2\omega A_0}$ as the arguments. *Bessel functions with an infinite number of arguments* are defined and employed.

13. It is pointed out as a remarkable fact that the trajectories of the existing planets of the Solar System are of the type quasi-elliptic but very near to the type of motion corresponding to small values of λ. This is closely related as a necessary consequence to the circumstance that the Newtonian kinetic energy at infinity is negative but very nearly zero and at the same time the mass of the Sun is very small so as to keep the relativistic trajectories with the least possible deviation from the Newtonian.

January, 1931.

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Notes added on March 15.

1. After I sent this manuscript to the press I learnt through a paper by Infeld, Phys. Zeits., **32** (1931) 110, that Synge had obtained before me a similar theorem to our Corollary in our Chapter I. But his proof and his method of attack are quite different from ours. Our Lemma is a generalisation of Synge's theorem. Synge, Math. Ann., **99** (1928) 738, especially p. 751.

2. Recently McCrea and McVittie have deduced the line element in a space with a globular cluster or a nebula at the origin of the co-ordinate under the assumption that the radius of the universe varies with the time according to Lemaître. If we put the cosmological term equal to zero and if the regions far from the origin are excluded from

our consideration, then this line element was proved to reduce to Schwarzschild's. Hence the foregoing discussion is also valid for the motion of a star near but outside a globular cluster or a nebula. McCrea and McVittie, M. N. **91** (1930) 128.

3. Milne has just proposed a new theory on the structure of the stars. If a star is highly collapsed in the sense of Milne, that is, if the outer envelope of perfect gas and the inner envelope of Fermi gas are both of the collapsed type and if the star is merely of a bare incompressible core, then it would be possible to find out in the vicinity of such a star an example of the interesting types of motion discussed in this paper. Milne, M. N., **91** (1930) 4.

4. When we take the cosmological term into our account, the behaviour of the motion at infinity in the above discussion is slightly modified. The infinity in the above treatment corresponds either to an infinity point or to a line at infinity according as we adopt the pseudo-elliptic world of de Sitter or the cylindrical world of Einstein. The projective properties are not modified even if we consider the cosmological term. *Cf.*, Klein, Gött. Nach. (1918) 394.

A Further Note.

According to Lewis the correspondence between a dynamical problem and its analogue in geometry is due to Hertz. Lewis obtained another mode of correspondence between these two when the potential function is not zero, on purpose of applying it to Schrödinger's equation in wave mechanics. T. Lewis, Phil. Mag. [vii] **11** (1931) 753.