exhibiting any loss in fluorescent power. When, however, they were exposed in sealed evacuated quartz tubes the fluorescent power was gradually and irrecoverably destroyed by ultra-violet light of the shorter wave-lengths.

The experiments on the fluorescence of aqueous solutions of æsculin contained in evacuated sealed glass tubes are of special importance, for they go to show that molecules of æsculin can be made to fluoresce strongly for hours without being destroyed when subjected to light of suitable wave-length. This result would seem to show that the view put forward by Perrin, namely, that the emission of fluorescent light by an organic substance is evidence of its destruction is not tenable generally.

The Propagation of Gravitational Waves.

By A. S. Eddington, F.R.S.

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1. The problem of the propagation of disturbances of the gravitational field was investigated by Einstein in 1916, and again in 1918.* It has usually been inferred from his discussion that a change in the distribution of matter produces gravitational effects which are propagated with the speed of light; but I think that Einstein really left the question of the speed of propagation rather indefinite. His analysis shows how the co-ordinates must be chosen if it is desired to represent the gravitational potentials as propagated with the speed of light; but there is nothing to indicate that the speed of light appears in the problem, except as the result of this arbitrary choice. So far as I know, the propagation of the absolute physical condition—the altered curvature of space-time—has not hitherto been discussed.

Weyl† has classified plane gravitational waves into three types, viz.: (1) longitudinal-longitudinal; (2) longitudinal-transverse; (3) transverse-transverse. The present investigation leads to the conclusion that transverse-transverse waves are propagated with the speed of light *in all systems of co-ordinates*. Waves of the first and second types have no fixed velocity—a result which rouses suspicion as to their objective existence. Einstein had also become suspicious of these waves (in so far as they occur in his special co-ordinate-system) for another reason, because he found that they convey no energy.

* 'Berlin Sitzungsberichte,' p. 688 (1916); p. 154 (1918).

† 'Raum, Zeit, Materie,' 4th edition, p. 228; English edition, p. 252.

They are not objective, and (like absolute velocity) are not detectable by any conceivable experiment. They are merely sinuosities in the co-ordinate-system, and the only speed of propagation relevant to them is "the speed of thought."

The purpose of Einstein's special choice of co-ordinates now becomes clear; they are chosen so as to compel these spurious waves to travel with the same speed as the genuine waves. He imposes conditions on the co-ordinates which bar out most of the spurious disturbances possible; but those which have the velocity of light take advantage of their close resemblance to the genuine waves and slip through the barrier. It is evidently a great convenience in analysis to have all waves, both genuine and spurious, travelling with one velocity; but it is liable to obscure physical ideas by mixing them up so completely. The chief new point in the present discussion is that when unrestricted co-ordinates are allowed the genuine waves continue to travel with the velocity of light and the spurious waves cease to have any fixed velocity.

Plane waves are a very special and artificial case of gravitational propagation, and in the second part of the paper I consider divergent waves. Although the equations of the theory are the same as those occurring in the propagation of sound waves, there is no propagation of gravitation uniformly in all directions like a spherical sound wave. To show the type of divergent waves which can exist, we develop in detail the solution for the waves produced by a spinning rod. We confirm Einstein's result that the rod will slowly lose energy by these waves. Spurious divergent waves can exist just as spurious plane waves can exist; but there is no longer a simple criterion for distinguishing them. The only case in which Weyl's three types of waves exist independently of one another is that of plane waves.

Our calculations show that there is one type of transverse-transverse waves which is inconsistent with the equations $G_{\mu\nu} = 0$. We have examined this case and find that such waves can actually exist, but must be accompanied by other physical manifestations not of a purely gravitational character. This one remaining mode of propagation of gravitational influence turns out to be propagation by light waves.

Plane Waves.

2. Take approximately Galilean co-ordinates and set

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu},$$

where $\delta_{\mu\nu}$ denotes the Galilean values and $h_{\mu\nu}$ the small deviations representing the passage of gravitational waves.

We consider plane waves proceeding with velocity V in the negative direction of the axis of x, so that the $h_{\mu\nu}$ are periodic functions of $(x+\mathrm{V}t)$ only. Writing as usual (x_1, x_2, x_3, x_4) for (x, y, z, t), the argument is $(x_1+\mathrm{V}x_4)$. Denoting differentiation with respect to this argument by an accent, we have

$$\frac{\partial^2 g_{\mu\nu}}{\partial x_1^2} = h_{\mu\nu}'', \qquad \frac{\partial^2 g_{\mu\nu}}{\partial x_1 \partial x_4} = \mathrm{V}h_{\mu\nu}'', \qquad \frac{\partial^2 g_{\mu\nu}}{\partial x_4^2} = \mathrm{V}^2 h_{\mu\nu}'', \tag{1}$$

and all other second derivatives are zero.

We consider waves of small amplitude, so that the square of the amplitude is neglected. The Riemann-Christoffel tensor is calculated from the formula

$$(\mu\rho\nu\sigma) = \mathrm{B}_{\mu\nu\sigma\rho} = \tfrac{1}{2}\left(\frac{\partial^2 g_{\mu\nu}}{\partial x_\sigma \partial x_\rho} + \frac{\partial^2 g_{\sigma\rho}}{\partial x_\mu \partial x_\nu} - \frac{\partial^2 g_{\mu\sigma}}{\partial x_\nu \partial x_\rho} - \frac{\partial^2 g_{\nu\rho}}{\partial x_\mu \partial x_\sigma}\right), \tag{2}$$

where the terms of the second order of small quantities (products of Christoffel symbols) have been omitted. The 21 different components are easily found from (1) and (2) as follows :—

$$\left.\begin{array}{lll}
(1212) = \tfrac{1}{2}h_{22}'' & (1224) = -\tfrac{1}{2}\mathrm{V}h_{22}'' & (2424) = \tfrac{1}{2}\mathrm{V}^2 h_{22}'' \\
(1213) = \tfrac{1}{2}h_{23}'' & (1234) = -\tfrac{1}{2}\mathrm{V}h_{23}'' = (1324) & (2434) = \tfrac{1}{2}\mathrm{V}^2 h_{23}'' \\
(1313) = \tfrac{1}{2}h_{33}'' & (1334) = -\tfrac{1}{2}\mathrm{V}h_{33}'' & (3434) = \tfrac{1}{2}\mathrm{V}^2 h_{33}'' \\
\multicolumn{3}{c}{(1223), (1323), (1423), (2323), (2324), (2334) = 0} \\
\multicolumn{3}{l}{(1424) = \tfrac{1}{2}\mathrm{V}^2 h_{12}'' - \tfrac{1}{2}\mathrm{V}h_{24}'' \qquad (1434) = \tfrac{1}{2}\mathrm{V}^2 h_{13}'' - \tfrac{1}{2}\mathrm{V}h_{34}''} \\
\multicolumn{3}{l}{(1214) = \tfrac{1}{2}h_{24}'' - \tfrac{1}{2}\mathrm{V}h_{12}'' \qquad (1314) = \tfrac{1}{2}h_{34}'' - \tfrac{1}{2}\mathrm{V}h_{13}''} \\
\multicolumn{3}{c}{(1414) = \tfrac{1}{2}\mathrm{V}^2 h_{11}'' - \mathrm{V}h_{14}'' + \tfrac{1}{2}h_{44}''}
\end{array}\right\} . \tag{3}$$

Einstein's tensor $\mathrm{G}_{\mu\nu}$ is given by

$$\mathrm{G}_{\mu\nu} = g^{\sigma\rho}\mathrm{B}_{\mu\nu\sigma\rho} = g^{\sigma\rho}(\mu\rho\nu\sigma) = \delta^{\sigma\rho}(\mu\rho\nu\sigma)$$

to the first order of small quantities. For example,

$$\begin{aligned}
\mathrm{G}_{23} &= \delta^{\sigma\rho}(2\rho 3\sigma) \\
&= -(2131) - (2232) - (2333) + (2434) \\
&= -(1213) - 0 - 0 + (2434)
\end{aligned}$$

by the antisymmetrical properties of the tensor. Treating the other components similarly, Einstein's equations are :

$$\left.\begin{array}{ll}
\mathrm{G}_{11} = -(1212) - (1313) + (1414) = 0 & \mathrm{G}_{13} = (1223) + (1434) = 0 \\
\mathrm{G}_{22} = -(1212) - (2323) + (2424) = 0 & \mathrm{G}_{14} = (1224) + (1334) = 0 \\
\mathrm{G}_{33} = -(1313) - (2323) + (3434) = 0 & \mathrm{G}_{23} = -(1213) + (2434) = 0 \\
\mathrm{G}_{44} = -(1414) - (2424) - (3434) = 0 & \mathrm{G}_{24} = -(1214) + (2334) = 0 \\
\mathrm{G}_{12} = -(1323) + (1424) \qquad\qquad = 0 & \mathrm{G}_{34} = -(1314) - (2324) = 0
\end{array}\right\} . \tag{4}$$

Substituting from (3), these become (omitting the common factor $\frac{1}{2}$)

$$\left.\begin{array}{ll} -(h_{22}''+h_{33}'')+V^2h_{11}''-2Vh_{14}''+h_{44}''=0 & V^2h_{13}''-Vh_{34}''=0 \\ -h_{22}''+V^2h_{22}''=0 & -V(h_{22}''+h_{33}'')=0 \\ -h_{33}''+V^2h_{33}''=0 & -h_{23}''+V^2h_{23}''=0 \\ -(V^2h_{11}''-2Vh_{14}''+h_{44}'')-V^2(h_{22}''+h_{33}'')=0 & -h_{24}''+Vh_{12}''=0 \\ V^2h_{12}''-Vh_{24}''=0 & -h_{34}''+Vh_{13}''=0 \end{array}\right\}. \quad (5)$$

These equations can be integrated, *i.e.*, the double accents can be removed. There are no constants of integration, since the $h_{\mu\nu}$ are periodic functions. The equations then reduce to the following seven conditions:—

$$h_{22}+h_{33}=0. \quad (6a)$$

$$(1-V^2)(h_{22}, h_{33}, h_{23})=0. \quad (6b)$$

$$h_{24}=Vh_{12}\,;\ h_{34}=Vh_{13}. \quad (6c)$$

$$h_{44}-2Vh_{14}+V^2h_{11}=0. \quad (6d)$$

3. We can separate the coefficients of the disturbance into three groups, viz.:

Transverse-transverse	h_{22}, h_{33}, h_{23}.
Longitudinal-transverse	$h_{12}, h_{13}, h_{24}, h_{34}$.
Longitudinal-longitudinal	h_{11}, h_{14}, h_{44}.

It will be seen that the three groups represent disturbances which are propagated quite independently of one another; for example, the presence or absence of longitudinal-longitudinal waves makes no difference to the conditions (6c) which have to be fulfilled by the coefficients of the longitudinal-transverse waves. I shall indicate the three classes of waves by the initials TT, LT, LL respectively.

For TT waves, h_{22}, h_{23}, h_{33} cannot all vanish; hence by (6b)

$$1-V^2=0.$$

Accordingly, TT waves are propagated with velocity unity, *i.e.*, the velocity of light.

For LL and LT waves, h_{22}, h_{23}, h_{33} are zero, and there is no independent equation determining V. The value of V found from (6c) and (6d) depends on the coefficients of the disturbance, and has no tendency to approximate to the velocity of light.

With a view to discriminating between genuine disturbances of space-time and spurious waves introduced by using sinuous co-ordinate systems, we return to the consideration of the Riemann-Christoffel tensor (3). In consequence of the conditions (6c) and (6d), the last five components given in the Table (3) all vanish. The surviving components contain only h_{22}'', h_{23}'', h_{33}''. Accordingly, the Riemann-Christoffel tensor depends solely on the TT waves.

For LL or LT waves the Riemann-Christoffel tensor vanishes, so that space-time is flat and the supposed disturbance is an analytical fiction. The waves will disappear altogether if a suitable change of co-ordinates is made.

The principal invariant of empty space $B_{\mu\nu\sigma\rho}B^{\mu\nu\sigma\rho}$ being composed entirely of the TT coefficients h_{22}'', h_{23}'', h_{33}'', will be propagated with their velocity, viz., the velocity of light. This is the answer to the main question which we have set before ourselves—whether the absolute gravitational influence, independent of co-ordinate systems, is propagated with this speed. This was perhaps a foregone conclusion. If we have an absolute gravitational disturbance which has entirely detached itself from the material system which originated it, only one thing can happen to it. It cannot stay at rest, since there is no absolute rest. It must travel; and since there exists only one speed which is independent of frames of reference, it has no choice but to accept that speed. Our direct calculation verifies this prediction.

4. Since $h_{22}+h_{33} = 0$, typical* plane TT waves are represented by

$$ds^2 = \text{steady value} + (\mathrm{A}\,dy^2 - \mathrm{A}\,dz^2 + 2\mathrm{H}\,dy\,dz)\sin 2\pi\,(x+t)/\lambda.$$

By re-orientating the axes of y and z, the term in H can be made to disappear, so that

$$ds^2 = \text{steady value} + (dy^2 - dz^2)\,\mathrm{A}\sin 2\pi\,(x+t)/\lambda.$$

It is of interest to examine what kind of change in the distribution of matter is needed to originate these waves. Let the source be a thin plate lying between $x = \pm\epsilon$ and sending out waves symmetrically in both directions, so that

$$ds^2 = \text{const.} + (dy^2 - dz^2)\,\mathrm{A}\sin 2\pi\,(t+x+\epsilon)/\lambda \qquad (x < -\epsilon).$$

$$ds^2 = \text{const.} + (dy^2 - dz^2)\,\mathrm{A}\sin 2\pi\,(t-x+\epsilon)/\lambda \qquad (x > +\epsilon).$$

These can be connected continuously by taking within the plate

$$ds^2 = \text{const.} + (dy^2 - dz^2)\,\mathrm{A}\sin 2\pi\,t/\lambda \qquad (-\epsilon < x < +\epsilon).$$

In this region the tensor $G_{\mu\nu}$ will not vanish. Our previous calculations of $G_{\mu\nu}$ in (5) can be adapted by noting that $h_{\mu\nu}$ is now a function of t only, and we have only to retain those terms coming from double differentiation with respect to t; these are recognised by the presence of the coefficient V^2.

* The most general type is compounded from two of these systems since H need not have the same phase as A.

Thus :—

$$G_{22} = \tfrac{1}{2}h_{22}'' = -\frac{2\pi^2}{\lambda^2} A \sin\frac{2\pi t}{\lambda},$$

$$G_{33} = \tfrac{1}{2}h_{33}'' = \frac{2\pi^2}{\lambda^2} A \sin\frac{2\pi t}{\lambda},$$

and all other components are zero. We have also

$$G = \delta^{22}G_{22} + \delta^{33}G_{33} = 0.$$

The material energy-tensor is given by

$$-8\pi T_{\mu\nu} = G_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}G. \tag{7}$$

Hence

$$T_{22} = -T_{33} = \frac{\pi}{4\lambda^2} A \sin\frac{2\pi t}{\lambda},$$

and the other components vanish.*

The components T_{22} and T_{33} are the stresses p_{yy} and p_{zz} in the plate, so that a suitable source of TT waves is a periodic stress alternating between two directions at right angles in the plane of the plate. An equivalent source would be a circular motion of the plate in its own plane.

It may seem remarkable that the only genuine gravitational waves arise from very secondary disturbances in the distribution of matter. Moreover, when (as usual) the velocity of the molecules or of the plate as a whole is small compared with the velocity of light, the components T_{22} and T_{33} are of the second order in this small quantity. The principal component, T_{44}, and the components of the first order, T_{14}, etc., do not excite waves by their oscillations. This is at least partly attributable to the very artificial circumstances of production of *plane* waves. If it seems to us that a backward and forward motion of the plate (variation of T_{14}) ought to produce a more considerable disturbance than mere molecular interchanges without mass-motion, we must reflect that, even on the Newtonian theory of attractions, the to-and-fro motion of an *infinite* plate leaves the gravitational field undisturbed. A small plate would produce waves. Again, we might expect powerful waves to be produced by the alternate creation and destruction of mass (variations of T_{44}); but nature, having made no provision for the propagation of the corresponding disturbance, thereby automatically prevents the construction of such a source.

The problem of plane waves is too special and artificial to illustrate fully the ordinary propagation of gravitational potential. We therefore proceed to consider divergent waves.

* This refers to the periodic part of the energy tensor. There must be a steady distribution superposed corresponding to the matter which is executing the small oscillations.

Divergent Waves.

5. The general problem of gravitational disturbance due to a prescribed distribution and motion of matter can be treated in the following way. The $G_{\mu\nu}$ vanish in empty space, and for an assigned density, momentum, etc., of the matter they are given by equation (7); consequently in this problem the $G_{\mu\nu}$ have prescribed values at every point. We shall denote these prescribed values by $[G_{\mu\nu}]$. Consider a set of values of $h_{\mu\nu}$ (vanishing at infinity) obtained by solving the equations

$$\Box\, h_{\mu\nu} = 2[G_{\mu\nu}], \tag{8}$$

where the operator $\Box$ denotes $\partial^2/\partial t^2 - \nabla^2$. It can be deduced* from (8) that

$$\Box \left(\frac{\partial}{\partial x_\alpha} h_\mu{}^\alpha - \tfrac{1}{2}\frac{\partial h}{\partial x_\mu}\right) = 0, \tag{9}$$

where $h_\mu{}^\alpha = \delta^{\alpha\sigma} h_{\mu\sigma}$ and $h = h_\alpha{}^\alpha$. This equation is shown to hold everywhere, in matter as well as empty space, provided only that the assigned distribution $[G_{\mu\nu}]$ is consistent with the laws of conservation of energy and momentum. The integral of (9) (with $h_{\mu\nu}$ vanishing at infinity) is

$$\frac{\partial}{\partial x_\alpha} h_\mu{}^\alpha - \tfrac{1}{2}\frac{\partial h}{\partial x_\mu} = 0. \tag{10}$$

It can now be shown that (8) and (10) together lead to

$$G_{\mu\nu} = [G_{\mu\nu}]. \tag{11}$$

Accordingly the solution of (8) gives values of $h_{\mu\nu}$ which will satisfy the prescribed conditions, and constitutes a solution of the actual problem. Now (8) is the usual equation of wave propagation with unit velocity from sources of strength $2[G_{\mu\nu}]$, and is solved by retarded potentials, viz.,

$$h_{\mu\nu} = \frac{1}{2\pi}\iiint [G_{\mu\nu}]_{t-r} \cdot \frac{d\xi\, d\eta\, d\zeta}{r}, \tag{12}$$

where r is the distance from (ξ, η, ζ) to (x, y, z).

The foregoing is essentially Einstein's treatment rearranged to show the rigour of the argument; and (12) constitutes his solution of the problem. It is correct only to the first order of small quantities. It leads to a definite co-ordinate system. Other solutions having a different analytical appearance could be obtained by performing co-ordinate transformations on the $h_{\mu\nu}$ without infringing the condition that they must be small quantities.

Consider the disturbance due to a single source, that is to say a material

* For the intermediate steps in the analysis, see 'Espace, Temps et Gravitation,' Supplement, §43, or my 'Mathematical Theory of Relativity,' §57.

system confined to a small region at the origin. Then $G_{\mu\nu} = 0$ except in this region. Let

$$\{G_{\mu\nu}\} = \iiint G_{\mu\nu}\, d\xi\, d\eta\, d\zeta$$

taken over the source. Treating r as constant over the source, the solution becomes

$$h_{\mu\nu} = \frac{1}{2\pi r}\{G_{\mu\nu}\}_{t-r}. \tag{13}$$

Thus, for a periodic source, $\{G_{\mu\nu}\} = A_{\mu\nu}\, e^{ipt}$, the solution is

$$h_{\mu\nu} = \frac{A_{\mu\nu}}{2\pi} \cdot \frac{e^{ip(t-r)}}{r}, \tag{14}$$

in agreement with the usual formula for spherical waves in the theory of sound and of electro-magnetism.

I am not sure that any writer has explicitly stated that gravitation is propagated isotropically by spherical waves of this kind; but I think that many like myself must have regarded it as an inference from Einstein's equation (8) which was two obvious to need mentioning. It was a great surprise to me to find that the solution (14) *does not satisfy the law of gravitation.*

To check the solution we shall calculate the component G_{23} at a point on the axis of x_1. When (as here) the $h_{\mu\nu}$ are functions of r and t only, the non-vanishing second derivatives are

$$\frac{\partial^2}{\partial x_1^2},\ \frac{\partial^2}{\partial x_2^2},\ \frac{\partial^2}{\partial x_3^2},\ \frac{\partial^2}{\partial x_4^2},\ \frac{\partial^2}{\partial x_1 \partial x_4}.$$

The other five vanish on the x_1-axis. Hence by (2) we find

$$(1213) = \tfrac{1}{2}\frac{\partial^2 h_{23}}{\partial x_1^2} \qquad (2434) = \tfrac{1}{2}\frac{\partial^2 h_{23}}{\partial x_4^2},$$

so that by (4) the condition $G_{23} = 0$ becomes

$$\tfrac{1}{2}\left(\frac{\partial^2 h_{23}}{\partial x_4^2} - \frac{\partial^2 h_{23}}{\partial x_1^2}\right) = 0,$$

which is the equation for plane waves and is not satisfied by (14) or even by the more general form (13).

We can calculate similarly the conditions $G_{12} = 0$, $G_{24} = 0$, and with a little more trouble isolate the component h_{24}. It appears that it also has to satisfy conditions inconsistent with (13) or (14). It follows by symmetry that the solution also fails for h_{12}, h_{13}, h_{14}, h_{24}.

I have not isolated the components h_{11}, h_{22}, h_{33}, h_{44}, but we shall see later that the periodic solution fails for these also. It seems that the only case in which (13) satisfies the law of gravitation is the static case of an undisturbed particle. Thus although the equations of propagation appear to be exactly analogous to the equations of the theory of sound, there is a remarkable difference in the type of propagation since isotropic spherical waves of gravitation cannot occur.

6. In our discussion of plane waves we did not explicity exclude waves of supernatural origin; it was sufficient that the waves themselves should obey the law of gravitation. But in deducing Einstein's solution in § 5 it was necessary to insert the condition that the sources obeyed the conservation of energy and momentum. There can be no doubt that the general solution (12) is correct, and the cause of its apparent failure must be that we have been applying it to a kind of source which cannot exist consistently with the laws of conservation. There cannot be a *simple* source of gravitational waves.

At first sight this conclusion seems contrary to common experience; we have no difficulty in constructing a simple source of G_{22}, G_{23}, G_{33} by spinning a rod in the yz-plane. But in the spinning rod the simple source for these components is accompanied by a *doublet* source of momentum G_{24}, G_{34}. The whole disturbance is therefore not a simple source, and the complete system of waves is not of the symmetrical type (14).

It can be deduced generally from equation (10) that simple waves of type (14) for any component of $h_{\mu\nu}$ must always be accompanied by doublet waves for some other component.

The reason for the breakdown of the analogy between sound waves and gravitational waves is now clear. Sound waves satisfy $\Box\phi = 0$, unconditionally; gravitational waves satisfy $\Box g_{\mu\nu} = 0$, but not unconditionally. And, owing to this condition, the most common solution for sound waves is not a solution for gravitational waves. The condition is superfluous in practical problems, because it is the condition that the sources shall be such as can occur in practical problems; but it none the less limits the number of admissible solutions.

I had hoped to generalise the results for plane waves by making a corresponding discussion for simple spherical waves; but we now see that simple spherical waves do not exist. I turn, therefore, to the consideration of spherical waves of a more complex type, which are certainly possible since they emanate from a material source which can actually be constructed.

We shall calculate the complete solution for a rod spinning in the plane of yz. Let $2a$ be the length, σ the mass per unit length, and ω the angular velocity. Choose an instant when the length of the rod is along the y-axis. The velocity, being in the z direction, will contribute to the component T_{33} of the energy-tensor. The integrated amount is $\{T_{33}\} = \frac{2}{3}\sigma\omega^2a^3$. There will necessarily be a tension in the spinning rod; this is in the y direction and contributes to T_{22}. The integrated amount is found to be $\{T_{22}\} = -\frac{2}{3}\sigma\omega^2a^3$.

For any other position of the rod the new values can be calculated by the tensor-law of invariance

$$T_{22}dy^2+2T_{23}dy\,dz+T_{33}dz^2 = T_{22}'\,dy'^2+2T_{23}'dy'\,dz'+T_{33}'\,dz'^2.$$

Setting $dy = dy'\cos\omega t - dz'\sin\omega t$, $dz = dy'\sin\omega t + dz'\cos\omega t$, the new coefficients when the rod makes an angle ωt with the axis of y are found to be

$$\{T_{22}\} = -\{T_{33}\} = -\tfrac{2}{3}\sigma\omega^2 a^3\cos 2\omega t, \quad \{T_{23}\} = -\tfrac{2}{3}\sigma\omega^2 a^3\sin 2\omega t.$$

Let

$$\gamma_{\mu\nu} = h_{\mu\nu}-\tfrac{1}{2}\delta_{\mu\nu}h.$$

Then by (7) and (14) we obtain

$$\gamma_{22} = f \qquad \gamma_{23} = -if \qquad \gamma_{33} = i^2 f, \tag{15a}$$

where

$$f = \tfrac{2}{3}\sigma p^2 a^3\frac{e^{ip(t-r)}}{r}, \qquad p = 2\omega.$$

As already explained these simple waves from the stress components must be accompanied by doublet waves from the momentum components, which in turn require quadruplet waves from the energy-component. We can calculate these waves most easily from equation (10) which is equivalent to $\partial\gamma_\mu{}^\alpha/\partial x_\alpha = 0$. Hence

$$\frac{\partial\gamma_{24}}{\partial t} = \frac{\partial\gamma_{22}}{\partial y}+\frac{\partial\gamma_{23}}{\partial z}, \qquad \frac{\partial\gamma_{34}}{\partial t} = \frac{\partial\gamma_{23}}{\partial y}+\frac{\partial\gamma_{33}}{\partial z}, \qquad \frac{\partial\gamma_{44}}{\partial t} = \frac{\partial\gamma_{24}}{\partial y}+\frac{\partial\gamma_{34}}{\partial z}.$$

These give by (15a)

$$\left.\begin{aligned} \gamma_{24} &= \frac{1}{ip}\left(\frac{\partial}{\partial y}-i\frac{\partial}{\partial z}\right)f \\ \gamma_{34} &= -\frac{1}{p}\left(\frac{\partial}{\partial y}-i\frac{\partial}{\partial z}\right)f \\ \gamma_{44} &= -\frac{1}{p^2}\left(\frac{\partial}{\partial y}-i\frac{\partial}{\partial z}\right)^2 f \\ \gamma &= \gamma_{44} = -h \end{aligned}\right\} \tag{15b}$$

If we write

$$u = \frac{1}{ip}\left(\frac{\partial}{\partial y}-i\frac{\partial}{\partial z}\right)$$

the complete set of values of $h_{\mu\nu}$ is given by the scheme

$$\begin{array}{llll} h_{\mu\nu} = \tfrac{1}{2}u^2 & 0 & 0 & 0 \\ 0 & 1+\tfrac{1}{2}u^2 & -i & u \\ 0 & -i & -1+\tfrac{1}{2}u^2 & -iu \\ 0 & u & -iu & \tfrac{1}{2}u^2 \end{array}$$

all operating upon f. These values constitute the solution of the problem. They satisfy (to the first order) the law of gravitation in empty space because

(8) and (10) are both satisfied. Moreover, the values of $\gamma_{22}, \gamma_{23}, \gamma_{33}$ correspond to the spinning rod; and since the values of $\gamma_{24}, \gamma_{34}, \gamma_{44}$ depend on these uniquely, they must also correspond—if the existence of a spinning rod is not inconsistent with the laws of nature.

Divergent waves can be created analytically by transforming the co-ordinate-system so that many other expressions for the solution may be given. We can, for example, remove the "momentum" waves h_{24}, h_{34}, but the other waves then become more complex. Make the transformation

$$x_\alpha' = \delta_\mu^\alpha x_\mu - \delta^{\mu\alpha}\xi^\mu,$$

where ξ^μ represents arbitrary functions of the co-ordinates of the first order of small quantities. From the transformation law of $g_{\mu\nu}$, we find

$$h_{\mu\nu}' = h_{\mu\nu} + \frac{\partial \xi^\mu}{\partial x_\nu} + \frac{\partial \xi^\nu}{\partial x_\mu}.$$

If

$$\xi^1 = 0, \qquad \xi^2 = \frac{1}{p^2}\left(\frac{\partial}{\partial y} - i\frac{\partial}{\partial x}\right)f, \qquad \xi^3 = -\frac{i}{p^2}\left(\frac{\partial}{\partial y} - i\frac{\partial}{\partial z}\right)f, \qquad \xi^4 = 0,$$

we find that $\gamma_{24}', \gamma_{34}', \gamma_{44}'$ disappear, but quadruplet waves are introduced into all the other components. Another useful co-ordinate-system is obtained by making half the above transformation. This has the great advantage that it makes h vanish, so that $\gamma_{\mu\nu} = h_{\mu\nu}$, and many of the formulæ of the theory are greatly simplified. The volume of any four-dimensional region is now undisturbed by the waves.

The chief point of practical interest in the problem of the spinning rod is the question whether its energy is gradually carried away by the gravitational waves which are created, so that the rotation would slow down and ultimately stop of its own accord. We shall use our solution to examine this question, which has also been treated more generally by Einstein. We agree with his conclusion that the energy will actually be radiated away, at a very slow rate.

The expression for the pseudo-energy-tensor of the gravitational field, correct to the *second* order, is*

$$32\pi t_{\mu\nu} = \left(\frac{\partial\gamma_{\alpha\beta}}{\partial x_\mu}\frac{\partial\gamma^{\alpha\beta}}{\partial x_\nu} - \tfrac{1}{2}\frac{\partial\gamma}{\partial x_\mu}\frac{\partial\gamma}{\partial x_\nu}\right) - \tfrac{1}{2}\delta_{\mu\nu}\left(\frac{\partial\gamma_{\alpha\beta}}{\partial x_\sigma}\frac{\partial\gamma^{\alpha\beta}}{\partial x^\sigma} - \tfrac{1}{2}\frac{\partial\gamma}{\partial x_\sigma}\frac{\partial\gamma}{\partial x^\sigma}\right), \qquad (16)$$

where $\gamma^{\alpha\beta} = -\gamma_{\alpha\beta}$ for the momentum components and $+\gamma_{\alpha\beta}$ for the other components.

$x^\sigma = -x_\sigma$ for the space components and $+x_\sigma$ for the time component.

* This is Einstein's expression (second paper, p. 158), modified because Einstein employs imaginary time. It applies to his special system of co-ordinates employed in our solution (15a, b), and must not be used for any other system.

The radial component of the flow of energy per unit area per unit time (Poynting vector) is obtained by resolving (t_4^1, t_4^2, t_4^3) along the radius. Thus :—

$$t_4^r = t_4^1 \frac{\partial x_1}{\partial r} + t_4^2 \frac{\partial x_2}{\partial r} + t_4^3 \frac{\partial x_3}{\partial r},$$

so that by (16)

$$32\pi t_4^r = -\left(\frac{\partial \gamma_{\alpha\beta}}{\partial t}\frac{\partial \gamma^{\alpha\beta}}{\partial r} - \tfrac{1}{2}\frac{\partial \gamma}{\partial t}\frac{\partial \gamma}{\partial r}\right)$$

which in our problem simplifies to

$$= -\left(\frac{\partial \gamma_{22}}{\partial t}\frac{\partial \gamma_{22}}{\partial r} + 2\frac{\partial \gamma_{23}}{\partial t}\frac{\partial \gamma_{23}}{\partial r} + \frac{\partial \gamma_{33}}{\partial t}\frac{\partial \gamma_{33}}{\partial r} - 2\frac{\partial \gamma_{24}}{\partial t}\frac{\partial \gamma_{24}}{\partial r} - 2\frac{\partial \gamma_{34}}{\partial t}\frac{\partial \gamma_{34}}{\partial r} + \tfrac{1}{2}\frac{\partial \gamma_{44}}{\partial t}\frac{\partial \gamma_{44}}{\partial r}\right). \tag{17}$$

To obtain the total flow across an infinite sphere, we have to evaluate this expression to the order $1/r^2$; each separate factor is therefore only required to the order $1/r$. Accordingly, ($15a$, b) becomes

$$\gamma_{22} = \mathrm{A}\cos p(t-r), \qquad \gamma_{23} = \mathrm{A}\sin p(t-r), \qquad \gamma_{33} = -\mathrm{A}\cos p(t-r),$$

$$\gamma_{24} = \mathrm{A}\left(-\frac{y}{r}\cos p(t-r) - \frac{z}{r}\sin p(t-r)\right),$$

$$\gamma_{34} = \mathrm{A}\left(-\frac{y}{r}\sin p(t-r) + \frac{z}{r}\cos p(t-r)\right),$$

$$\gamma_{44} = \mathrm{A}\left(\frac{y^2-z^2}{r^2}\cos p(t-r) + \frac{2yz}{r^2}\sin p(t-r)\right),$$

where $\mathrm{A} = \frac{2}{3}\sigma p^2 a^3/r$, and can be treated as constant in differentiating.

We find

$$32\pi r^2 t_4^r = (\tfrac{2}{3}\sigma p^3 a^3)^2\left[\frac{2x^2}{r^2} + \tfrac{1}{4}\frac{(y^2+z^2)^2}{r^4} + \frac{y^4+z^4-6y^2z^2}{4r^4}\cos 2p(t-r) + \frac{yz(y^2-z^2)}{r^4}\sin 2p(t-r)\right]. \tag{18}$$

Taking the average values over the whole sphere, the periodic terms are found to vanish, and the total radiation, $4\pi r^2 t_4^r$, is at the steady rate

$$\tfrac{1}{10}(\tfrac{2}{3}\sigma p^3 a^3)^2,$$

or, if I is the moment of inertia of the rod, the result is

$$\tfrac{32}{5}\mathrm{I}^2\omega^6. \tag{19}$$

I find that Einstein's general formula (*loc. cit.*, equation (30)) applied to our problem gives the result $\frac{16}{5}\mathrm{I}^2\omega^6$. The discrepancy is due to a numerical slip in one or other investigation, and is not of much importance.*

* I think that the factor $\frac{1}{2}$ was introduced by Einstein in order to eliminate the energy of the imaginary waves contained in the expression e^{ipt}; and it should have been dropped in a formula where imaginary waves are not introduced.

If C.G.S. units are used, we must multiply the expression (19) by $2{\cdot}7{\cdot}10^{-60}$. For example, a rod of mass 1 kgrm., length 2 metres, making 50 revolutions per second, is found to lose $3{\cdot}10^{-35}$ of its energy of rotation in a year.

The radiation depends on the asymmetry of the source; a circular disc spinning in its own plane would not lose any energy. The formula (19) would evidently apply to other linear distributions, such as two balls connected by a string. We might be tempted to apply it to pairs of masses not connected by material tension, *e.g.*, a hydrogen atom or a double star. For a hydrogen atom the rate of decay is much the same as for the rod, *i.e.*, of the order 10^{-35} per year; for a double-star system it is rather larger, about 10^{-20} per year. It is rather curious that the difference between these two systems at the opposite extremes of magnitude-scale should not be greater. But both applications are probably illegitimate—the atom, because it is complicated by quantum conditions outside our analysis; the star, because the tension (being now a gravitational force) is limited to a small quantity of the first order, and the problem is thus carried out of range of our approximation. But it seems likely that the radiation (if any) will not exceed that given by (19).

There is clearly no practical objection to the existence of this small radiation from rotating systems, and I can see no theoretical reason for not admitting it.

According to (17) the momentum components gave an inflow of energy which neutralises a considerable part of the outflow due to other components. Also, by (18), the flow of energy is mainly along the axis of rotation of the rod, being eight times more intense along the axis than at right angles to it. But these results have no physical interpretation. The various components of the potential have no separable existence, and the Poynting vector is not a true vector. As Einstein has pointed out, the investigation of this paragraph gives the total loss of energy of the material system correctly, but the intermediate steps are merely analytical. The lost energy is not localisable anywhere.

We found that for plane waves the genuine waves arise from the stress-condition of the material source, and we expressed surprise that so inconspicuous a feature of material distribution should be the occasion of them. Our study of divergent waves confirms this. Apparently, the only originating sources (as distinct from scattering sources) are typified by the spinning rod or by an oscillator (*e.g.*, two masses connected by a spring). We cannot disturb the world by creating mass nor by transporting mass within the same source, since the laws of mass and momentum render these processes undisturbable. The simplest interference we can make is to transport momentum, and that is the process represented by the stress-components.

Electromagnetic Gravitational Waves.

8. The three coefficients for plane TT waves may be grouped as follows:—

$$h_{23}, \qquad h_{22}-h_{33}, \qquad h_{22}+h_{33}.$$

The propagation of the first two has already been studied; but according to $(6a)$ propagation of the third group is impossible.

We shall now show that waves of the third kind can exist in space which, although not empty in the strictest sense, is empty of all matter. They are in fact the ordinary light waves. It is of interest to note that this electromagnetic mode of propagation of gravitational influence fits into the only gap left unfilled in the previous discussion of plane gravitational waves.

Plane polarised electromagnetic waves travelling in the negative direction of x are represented by the electromagnetic potential.

$$\kappa_2 = \mathrm{A}\sin p(t+x) = \mathrm{A}\sin p(x_4+x_1),$$

which gives for the electromagnetic force

$$\mathrm{F}_{21} = \frac{\partial \kappa_2}{\partial x_1} = p\mathrm{A}\cos p(t+x) = \mathrm{F}^{21},$$

$$\mathrm{F}_{24} = \frac{\partial \kappa_2}{\partial x_4} = p\mathrm{A}\cos p(t+x) = -\mathrm{F}^{24}.$$

From which, $\mathrm{F}_{\alpha\beta}\mathrm{F}^{\alpha\beta} = 0$, and the energy-tensor $\mathrm{E}_\mu^\nu = -\mathrm{F}_{\mu\alpha}\mathrm{F}^{\nu\alpha}+\frac{1}{4}g_\mu^\nu\mathrm{F}_{\alpha\beta}\mathrm{F}^{\alpha\beta}$ becomes

$$\mathrm{E}_{11} = \mathrm{E}_{14} = \mathrm{E}_{44} = p^2\mathrm{A}^2\cos^2 p(t+x),$$

the other components being zero. Corresponding to equations (5), we now have $\mathrm{G}_{11} = -8\pi\mathrm{E}_{11}$, etc., instead of $\mathrm{G}_{11} = 0$. Picking out the three equations to be modified, it will be seen that $(6a)$ is replaced by

$$h_{22}''+h_{33}'' = 16\pi p^2\mathrm{A}^2\cos^2 p(t+x),$$

and the other conditions are unaltered. Thus the periodic part of $h_{22}+h_{33}$ is

$$h_{22}+h_{33} = -2\pi\mathrm{A}^2\cos 2p(t+x). \tag{20}$$

There is also an aperiodic part $4\pi p^2\mathrm{A}^2(t+x)^2$, indicating that the field is undergoing a cumulative change. This can only mean a secular change in the distribution of energy, etc., and, as is well known, waves of this kind do actually carry away energy from the source—not a small quantity of the second order negligible in our approximations as in the previous waves discussed, but a quantity of the first order.

Returning to the periodic waves (20), we notice that the gravitation potentials give no indication of the plane of polarisation of the waves. A varying electromagnetic potential κ_3 instead of κ_2 would produce just the same gravitational effect.

In order to determine h_{22} and h_{33} separately we have to find the non-electromagnetic wave $h_{22}-h_{33}$ emanating from the material source. The gravitational polarisation of light-waves thus depends on the disturbance of the uncharged material as well as on the electrical oscillations, and is quite unconnected with the electromagnetic polarisation.

When $h_{22}+h_{33}$ is not zero there exists a true Poynting vector E_4^1 representing a flow of true energy of the same order of magnitude as $h_{22}''+h_{33}''$. We shall examine whether there is not in addition a Poynting pseudo-vector t_4^1 of the second order of small quantities similar to that found in § 7. Equation (16) is not now applicable, but the general expression is (Eddington, 'Mathematical Theory of Relativity,' equations (59·4) and (58·52)).

$$16\pi t_\mu^\nu = -\frac{\partial}{\partial x_\mu}(g^{\alpha\beta}\sqrt{-g})[-\{\alpha,\beta,\nu\}+g_\alpha^\nu\{\beta\sigma,\sigma\}] \qquad (\mu\neq\nu).$$

When only h_{22} and h_{33} occur, this reduces to

$$\begin{aligned}16\pi t_4^1 &= \frac{\partial}{\partial x_4}(g^{22}\sqrt{-g})\,.\,\{22,1\}+\frac{\partial}{\partial x_4}(g^{33}\sqrt{-g})\,.\,\{33,1\} \\ &\qquad\qquad -\frac{\partial}{\partial x_4}(\delta^{11}\sqrt{-g})\,.\,\{1\sigma,\sigma\} \\ &= -\tfrac{1}{4}\frac{\partial}{\partial x_4}(h_{22}-h_{33})\,.\,\frac{\partial}{\partial x_1}(h_{22}-h_{33})-\tfrac{1}{4}\frac{\partial}{\partial x_4}(h_{22}+h_{33})\,.\,\frac{\partial}{\partial x_1}(h_{22}+h_{33}).\end{aligned}$$

Accordingly the waves $(h_{22}+h_{33})$ carry away gravitational energy just like the waves $(h_{22}-h_{33})$. Light-waves by reason of their gravitational properties reduce the energy of their source by an amount slightly greater than is given by purely electromagnetic calculations. The ratio of this second order energy to the first order energy is usually very small indeed.